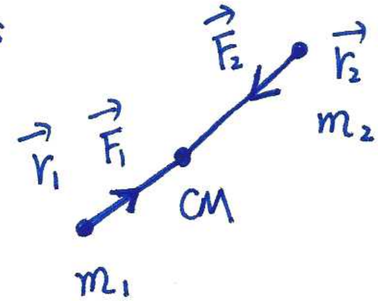
Lect 9 Kepler problem (I)

- CM and relative coordinates : reduced mass



$$\begin{cases} \vec{F}_1(|\vec{r}_1-\vec{r}_2|) = -\vec{F}_2(|\vec{r}_1-\vec{r}_2|) \\ m_1\ddot{\vec{r}}_1 = \vec{F}_1 & (1) \\ m_2\ddot{\vec{r}}_2 = \vec{F}_2 & (2) \end{cases}$$

$$(1)+(2)=0 \Rightarrow \boxed{\ddot{\vec{R}}=0} \quad \text{with} \quad \boxed{\vec{R} = \frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2}}$$ Center of mass coordinat

$$\frac{(1)}{m_1} - \frac{(2)}{m_2} \Rightarrow \ddot{\vec{r}} = \left(\frac{1}{m_1}+\frac{1}{m_2}\right)\vec{F}_1 \quad \text{where} \quad \boxed{\vec{r} = \vec{r}_1 - \vec{r}_2}$$ relative coordinate

$$\boxed{\mu\ddot{\vec{r}} = \vec{F}_1(|r|)} \qquad \mu = \frac{m_1 m_2}{m_1+m_2} \leftarrow \text{reduced mass.}$$

$\mu < m_1, m_2$

- seperation of center of mass motion and relative motion

- For the relative motion, it's reduced to a single mass point moving in a centeral force field $\vec{F}_1(|r|)$. The mass is replaced by μ.

- $T = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2}$ plug in $\begin{cases} \vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r} \\ \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r} \end{cases}$ with $M=m_1+$

$$= \frac{1}{2}m_1\left[\dot{\vec{R}}^2 + \left(\frac{m_2}{M}\right)^2\dot{\vec{r}}^2 + 2\dot{\vec{R}}\cdot\dot{\vec{r}}\frac{m_2}{M}\right]$$

$$+ \frac{1}{2}m_2\left[\dot{\vec{R}}^2 + \left(\frac{m_1}{M}\right)^2\dot{\vec{r}}^2 - 2\dot{\vec{R}}\cdot\dot{\vec{r}}\frac{m_1}{M}\right] = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2$$

- $E = T + \mathcal{U} = \frac{1}{2} M \dot{R}^2 + \underbrace{\frac{1}{2}\mu \dot{\vec{r}}^2 + \mathcal{U}(r)}_{\text{relative motion}}$

$\mathcal{L} = T - \mathcal{U} = \frac{1}{2} M \dot{R}^2 + \underbrace{\frac{1}{2}\mu \dot{r}^2 - \mathcal{U}(r)}$.

$\uparrow$

Lagrangian, we will learn later.

- $\vec{L}_{CM}$ in the CM frame, i.e, the frame that $\vec{R}$ is at rest.

$$\vec{L}_{CM} = (\vec{r}_1 - \vec{R}) \times m_1 (\dot{\vec{r}}_1 - \dot{\vec{R}}) + (\vec{r}_2 - \vec{R}) \times m_2 (\dot{\vec{r}}_2 - \dot{\vec{R}})$$

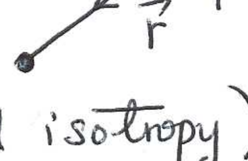
$$= \frac{m_2}{M} \vec{r} \times m_1 \frac{m_2}{M} \dot{\vec{r}} + \left(-\frac{m_1}{M}\vec{r}\right) \times m_2 \left(-\frac{m_1}{M}\right) \dot{\vec{r}}$$

$$= \frac{m_1 m_2}{M} \left(\frac{m_2 + m_1}{M}\right) \vec{r} \times \dot{\vec{r}} = \boxed{\mu \vec{r} \times \dot{\vec{r}} = \vec{L}_{CM}}$$

- Reduction to 1D motion

We have reduced the 2-body problem into a single body problem in 3D. Now let us further reduce it to 2D and to 1D motion. In the CM frame, $\vec{L}_{CM}$ is conserved!

The force passes the origin $\longrightarrow$ no torque.



(Angular momentum conservation due to spatial isotropy).

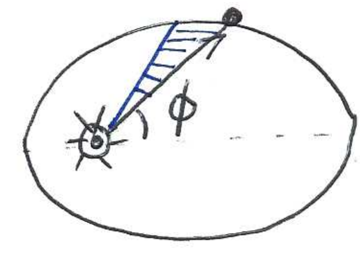
$$\frac{d}{dt} \vec{L}_{CM} = 0 \quad \Rightarrow \quad \vec{L}_{CM} \equiv \text{const vector}.$$

$\vec{L}_{cm}$ is perpendicular to the orbiatal plane $\Rightarrow$

$\boxed{\text{the motion is co-planar, say, in the xy-plane, and } \vec{L}_{cm} = l\hat{z}.}$

Then we use the equation of motion in the polar system

$$\begin{cases} F_r = \mu(\ddot{r} - r\dot{\phi}^2) & (1) \\ F_\phi = \mu(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = \frac{1}{r}\mu\frac{d}{dt}(r^2\dot{\phi}) & (2) \end{cases}$$



$F_\phi = 0 \Rightarrow \frac{d}{dt}[\mu r^2\dot{\phi}] = 0 \leftarrow$ This is Kepler's 2nd law.

Actually $\vec{L}_{cm} = l\hat{z} = \mu r\hat{r} \times \vec{v} = \mu r\hat{r} \times [\dot{r}\hat{r} + r\frac{d\hat{r}}{d\phi}\dot{\phi}]$

$= \mu r^2\dot{\phi}[\hat{r} \times \hat{\phi}] = \mu r^2\dot{\phi}\hat{z}$

$\Rightarrow \mu r^2\dot{\phi} = l \Rightarrow \dot{\phi} = \frac{l}{\mu r^2} \Rightarrow r\dot{\phi}^2 = \frac{l^2}{\mu^2 r^3}$

$\Rightarrow F_r = \mu\ddot{r} - \frac{l^2}{\mu r^3} \Rightarrow \boxed{\mu\ddot{r} = F_r + \frac{l^2}{\mu r^3}} \leftarrow$ effective 1D motion

Similarly, we can apply our previous knowledge on 1D motion to reduce it to 1st differential Eq.

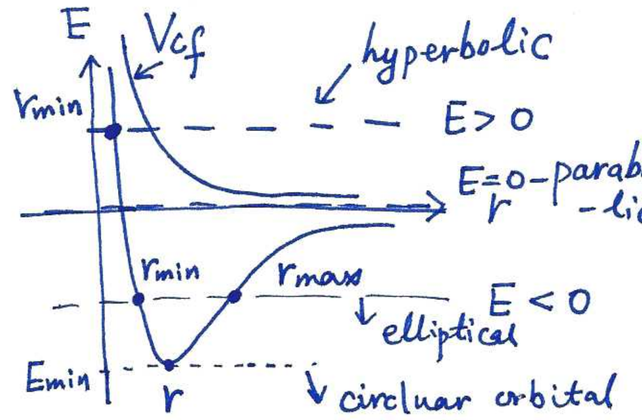
$$E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{U(r) + \frac{l^2}{2\mu r^2}}$$ where $U(r) = -\int_{r_0}^{r} F_r\,dr$

$$\boxed{E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)}$$

The effect of angular mometum is included by $\frac{l^2}{2\mu r^2} \triangleq V_{cf}(r)$

For Kepler problem $U(r) = -\dfrac{Gm_1m_2}{r} = -\dfrac{\gamma}{r}$ (where $\gamma = Gm_1m_2$)

$$U_{eff}(r) = -\frac{\gamma}{r} + \frac{l^2}{2\mu r^2}$$



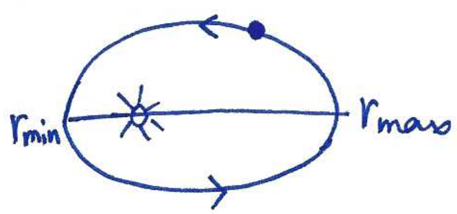
① $E < 0$: bound orbital

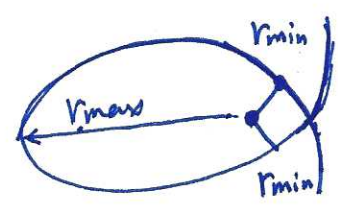
at E_{min}, the radial motion is at rest → circular motion

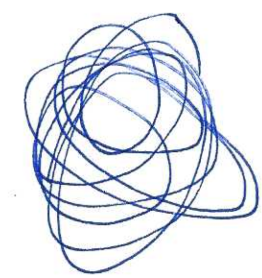
② $E = 0$ and $E > 0$ unbounded orbitals

What's special of $1/r^2$ - force field? — closed orbital at $E<0$

① The period of radial motion (bounce) is the same as angular period ϕ from $0 \sim 360°$.







② for general central force, the orbit may not be closed!

The ellipse may precess. The angular period isn't the same as the radial period.

- Solve the equation of orbit

$$\begin{cases} \mu \ddot{r} = F_r + \dfrac{\ell^2}{\mu r^3} & ① \longrightarrow \text{Solve } r(\phi) \\ \dot{\phi} = \dfrac{\ell}{\mu r^2} & ② \end{cases}$$

define $u = 1/r$ and we replace $\frac{d}{dt}$ by $\frac{d}{d\phi}$

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \frac{\ell}{\mu r^2}\frac{d}{d\phi} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}$$

$$\dot{r} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}\left(\frac{1}{u}\right) = -\frac{\ell}{\mu}\frac{du}{d\phi}$$

$$\ddot{r} = -\frac{\ell}{\mu}\frac{d}{dt}\frac{du}{d\phi} = -\frac{\ell}{\mu}\frac{\ell u^2}{\mu}\frac{d^2u}{d\phi^2} \Rightarrow -\frac{\ell^2u^2}{\mu^2}\frac{d^2u}{d\phi^2} = \frac{1}{\mu}F_r + \frac{\ell^2}{\mu^2}u^3$$

or $\boxed{\dfrac{d^2u}{d\phi^2} = -u(\phi) - \dfrac{\mu}{\ell^2u^2}F_r}$ plug in $F_r = -\dfrac{\gamma}{r^2} = -\gamma u^2$

$$\Rightarrow \frac{d^2u}{d\phi^2} = -u + \frac{\mu\gamma}{\ell^2}$$ ← inhomogenous 2nd order linear differential Eq

$$u = A\cos(\phi - \delta) + \frac{\mu\gamma}{\ell^2}$$ ← a special solution

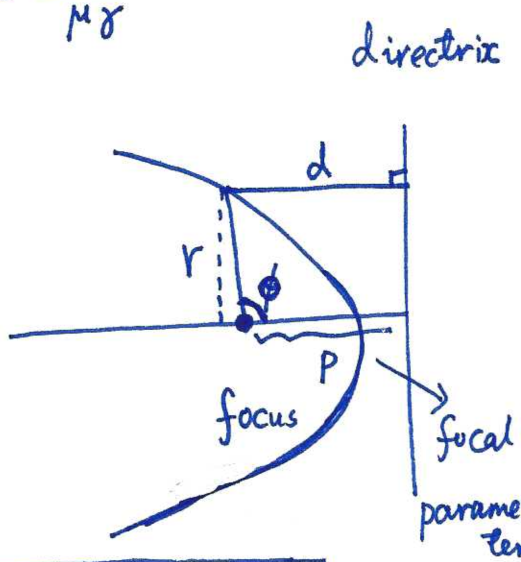
↑ Solution to the homogenous part

δ can be choose by choosing the x-axis along the angle δ-direction i.e. major axis.

$$\Rightarrow \frac{1}{r} = \frac{\mu\gamma}{\ell^2}\left[1 + e\cos\phi\right], \text{ where } e = \frac{A\ell^2}{\mu\gamma}$$

$$\Rightarrow \quad r(\phi) = \frac{c}{1 + e\cos\phi} \qquad \text{with} \quad \left\{ \begin{array}{l} c = \dfrac{\ell^2}{\mu\gamma} \\ e = \dfrac{A\ell^2}{\mu\gamma} \end{array} \right.$$

§ conic curves / sections

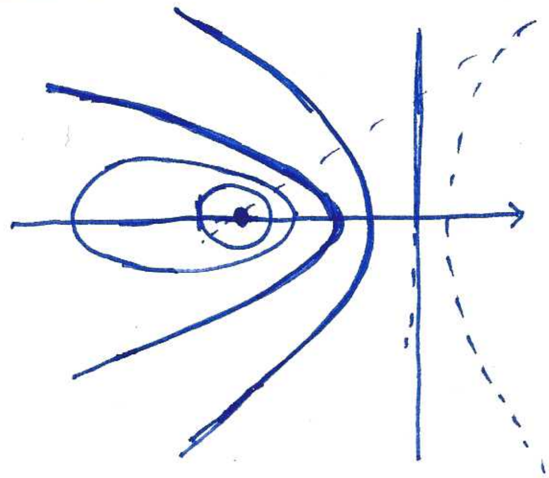


p: focal parameter

e: eccentricity

$$e = \frac{r}{d} \qquad \text{with} \quad d = p - r\cos\theta$$

$$\Rightarrow ed = ep - er\cos\theta = r \quad \Rightarrow \quad \boxed{r = \frac{ep}{1 + e\cos\phi}}$$



$0 < e < 1$ – ellipse

$e = 1$ – parabola

$e > 1$ hyperbola

change to Cartisican coordinate

$$r = ep - er\cos\phi \quad \leftarrow \quad r\cos\phi = x$$

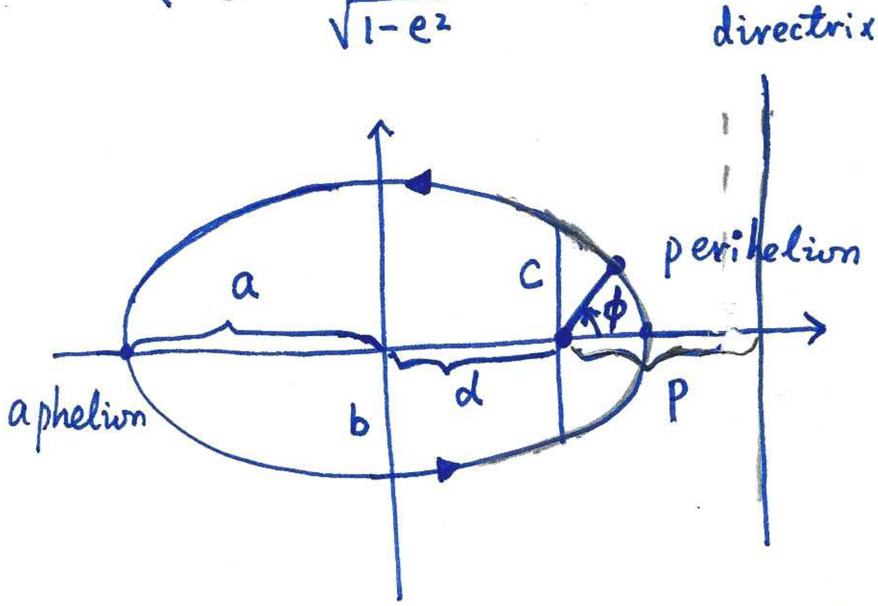
$$x^2 + y^2 = (ep)^2 + e^2x^2 - 2e^2p\,x$$

$$\boxed{(1-e^2)\left[x + \frac{e^2p}{1-e^2}\right]^2 + y^2 = \frac{e^2p^2}{1-e^2}}$$

for $0<e<1 \Rightarrow \dfrac{\left(x+\frac{e^2p}{1-e^2}\right)^2}{\left(\frac{ep}{1-e^2}\right)^2} + \dfrac{y^2}{\left(\frac{ep}{\sqrt{1-e^2}}\right)^2} = 1$

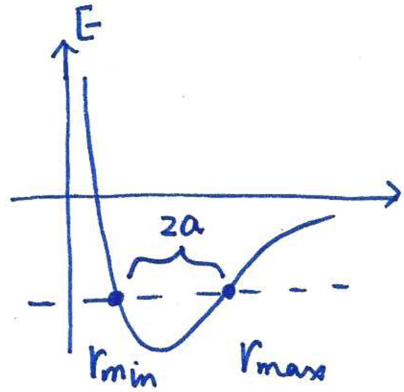
$$\Rightarrow \begin{cases} a = \dfrac{c}{1-e^2} \\ b = \dfrac{c}{\sqrt{1-e^2}} \end{cases} \qquad \begin{cases} c = ep = \dfrac{\ell^2}{\mu\gamma} \\ d = \dfrac{e^2p}{1-e^2} = ea \end{cases}$$

$$\begin{cases} e = \dfrac{A\ell^2}{\mu\gamma} \\ p = \dfrac{1}{A} \end{cases}$$



§ Express the orbit by using conserved quantities

- Energy: using the effective potential



$$U_{eff}(r) = -\frac{\gamma}{r} + \frac{\ell^2}{2\mu r^2}$$

$$r_{min} = \frac{c}{1+e} = \frac{\ell^2}{\mu\gamma(1+e)}$$

$$E = -\frac{\gamma}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2} = \frac{1}{2r_{min}}\left[\frac{\ell^2}{\mu r_{min}} - 2\gamma\right] = \left(\frac{\ell^2}{2\mu\gamma}\right)^{-1}(1+e)\,\gamma(e-1)$$

$$= \frac{\gamma^2\mu}{2\ell^2}(e^2-1) = \frac{-\gamma}{2a}$$

$$\boxed{a = \frac{\gamma}{-2E}}$$

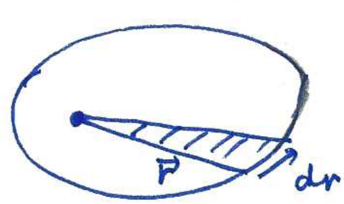
- the half-major axis "a" is only determined by the energy.

- The half Latus-rectum (Cord length) $\boxed{c = \frac{\ell^2}{\mu\gamma}}$ is only dertmined by the angular momentum

- $a = \dfrac{c}{1-e^2} \Rightarrow 1-e^2 = \dfrac{c}{a} = \dfrac{\ell^2}{\mu\gamma} \cdot \dfrac{-2E}{\gamma} \Rightarrow e = \sqrt{1 + \dfrac{2\ell^2 E}{\mu\gamma^2}}$

$$\frac{b^2}{a^2} = 1-e^2 \Rightarrow \frac{b^2}{a} = (1-e^2)\,a = c \Rightarrow b = \sqrt{\frac{\ell^2}{-2\mu E}}$$

§ Kepler's 3rd law



$$d\vec{A} = \frac{1}{2}\,\vec{r} \times d\vec{r} \Rightarrow$$

$$\frac{dA}{dt} = \frac{1}{2}\,\frac{\ell}{\mu}$$

The total area $A = \pi ab \Rightarrow \tau = \dfrac{A}{dA/dt} = \dfrac{2\pi ab\,\mu}{\ell}$

$$\Rightarrow \tau^2 = \frac{4\pi^2 a^2 a^2 (1-e^2)\,\mu^2}{\ell^2} = \frac{4\pi^2 a^3 c\,\mu^2}{\ell^2} = \frac{4\pi^2 a^3 \mu}{\gamma}$$

plug in $c = \dfrac{\ell^2}{\mu\gamma}$

$$\Rightarrow \boxed{\frac{\tau^2}{a^3} = \frac{4\pi^2\mu}{\gamma} = \frac{4\pi^2}{G M_{sun}}}$$

$$\gamma = G m_1 m_2 = G\mu\,(m_{sun} + M_{earth}) \approx G\mu\,M_{sun}$$