Lect 10. Canonical transformation

§1 Change variables in Lagrangian mechanics

$$\mathcal{L}(q_i, \dot{q}_i, t) \xrightarrow[Q_i = Q_i(q_1, \cdots q_n, t)]{} \tilde{\mathcal{L}}(Q_i, \dot{Q}_i, t) = \mathcal{L}(q_i(Q_j)\ \dot{q}_i(Q_j \dot{Q}_j, t)$$

From the least action principle

$$\delta \int_{t_a}^{t_b} dt\ \tilde{\mathcal{L}}(Q_i, \dot{Q}_i, t) = 0 \quad \text{with } \delta Q_i(t_a) = \delta Q_i(t_b) = 0$$

$$\Rightarrow \boxed{\frac{\partial \tilde{\mathcal{L}}}{\partial Q_i} = \frac{d}{dt}\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{Q}_i}} \text{ should still hold.}$$

We can also verify it explicitly

$$\frac{d}{dt}\left(\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{Q}_i}\right) = \frac{d}{dt}\left[\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{q}_\alpha}\frac{\partial \dot{q}_\alpha}{\partial \dot{Q}_i}\right)$$

$$q_\alpha = q_\alpha(Q_1, \cdots Q_n, t) \Rightarrow \dot{q}_\alpha = \frac{\partial q_\alpha}{\partial Q_i}\dot{Q}_i + \frac{\partial q_\alpha}{\partial t} \Rightarrow \frac{\partial \dot{q}_\alpha}{\partial \dot{Q}_i} = \frac{\partial q_\alpha}{\partial Q_i}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{Q}_i}\right) = \frac{d}{dt}\left(\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{q}_\alpha}\frac{\partial q_\alpha}{\partial Q_i}\right)$$

next

$$\frac{\partial \tilde{\mathcal{L}}}{\partial Q_i} = \frac{\partial \mathcal{L}}{\partial q_\alpha}\frac{\partial q_\alpha}{\partial Q_i} + \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\frac{\partial \dot{q}_\alpha}{\partial Q_i}$$

$$= \frac{\partial \mathcal{L}}{\partial q_\alpha}\frac{\partial q_\alpha}{\partial Q_i} + \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\left(\frac{\partial^2 q_\alpha}{\partial Q_j \partial Q_i}\dot{Q}_j + \frac{\partial^2 q_\alpha}{\partial Q_i \partial t}\right)$$

$$= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right)\frac{\partial q_\alpha}{\partial Q_i} + \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\frac{d}{dt}\left[\frac{\partial q_\alpha}{\partial Q_i}\right] = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\frac{\partial q_\alpha}{\partial Q_i}\right] = \frac{d}{dt}\left[\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{Q}_i}\right]$$

Canonical transformation for Hamiltonian

$H(q_i, p_i, t)$ $\qquad \dot{q}_i = \dfrac{\partial H}{\partial p_i}$, $\dot{p}_i = -\dfrac{\partial H}{\partial q_i}$. $\qquad (i=1,2,\cdots N)$

introduce transformations $\begin{cases} q_i = q_i(Q_j, \mathbb{P}_j, t) \\ p_i = p_i(Q_j, \mathbb{P}_j, t) \end{cases}$ or $\begin{cases} Q_i = Q_i(q_j, p_j, t) \\ \mathbb{P}_i = \mathbb{P}_i(q_j, p_j, t) \end{cases}$.

inverse

Question: what kind of transformation leaves the form of Hamilton Eqs invariant?

i.e. $\dot{Q}_i = \dfrac{\partial \tilde{H}}{\partial \mathbb{P}_i}$, $\dot{\mathbb{P}}_i = -\dfrac{\partial \tilde{H}}{\partial Q_i}$ $\qquad i=1,2,\cdots N$

and $\tilde{H}(Q, \mathbb{P}, t) = H(q(Q, \mathbb{P}), p(Q, \mathbb{P}), t)$.

For simplicity, we only work for the case of $N=1$. The generalization to arbitary N is straightforward. Let us check

$$\dot{Q} = \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p} = \frac{\partial Q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial H}{\partial q}$$

$$\dot{\mathbb{P}} = \frac{\partial \mathbb{P}}{\partial q}\dot{q} + \frac{\partial \mathbb{P}}{\partial p}\dot{p} = \frac{\partial \mathbb{P}}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial \mathbb{P}}{\partial p}\frac{\partial H}{\partial q}$$

$$\Rightarrow \begin{pmatrix} \dot{Q} \\ \dot{\mathbb{P}} \end{pmatrix} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial \mathbb{P}}{\partial q} & \frac{\partial \mathbb{P}}{\partial p} \end{pmatrix} \begin{pmatrix} & I \\ -I & \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}$$

on the hand, we want $Q, \mathbb{P}$ also satisfy the canonical Eq

$$\Rightarrow \begin{pmatrix} \dot{Q} \\ \dot{\mathbb{P}} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \tilde{H}}{\partial Q} \\ \frac{\partial \tilde{H}}{\partial \mathbb{P}} \end{pmatrix} = \begin{pmatrix} & I \\ -I & \end{pmatrix} \begin{pmatrix} \frac{\partial q}{\partial Q} & \frac{\partial p}{\partial Q} \\ \frac{\partial q}{\partial \mathbb{P}} & \frac{\partial p}{\partial \mathbb{P}} \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}$$

check

$$\left.\begin{aligned} \frac{\partial \tilde{H}}{\partial Q} &= \frac{\partial H}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial Q} \\ \frac{\partial \tilde{H}}{\partial \mathbb{P}} &= \frac{\partial H}{\partial q}\frac{\partial q}{\partial \mathbb{P}} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial \mathbb{P}} \end{aligned}\right\} \Rightarrow$$

$$\begin{pmatrix} \frac{\partial \tilde{H}}{\partial Q} \\ \frac{\partial \tilde{H}}{\partial \mathbb{P}} \end{pmatrix} = \begin{pmatrix} \frac{\partial q}{\partial Q} & \frac{\partial p}{\partial Q} \\ \frac{\partial q}{\partial \mathbb{P}} & \frac{\partial p}{\partial \mathbb{P}} \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}$$

$$\Rightarrow \underbrace{\begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial \mathbb{P}}{\partial q} & \frac{\partial \mathbb{P}}{\partial p} \end{pmatrix}}_{\text{define as } M} \begin{pmatrix} & I \\ -I & \end{pmatrix} = \begin{pmatrix} & I \\ -I & \end{pmatrix} \begin{pmatrix} \frac{\partial q}{\partial Q} & \frac{\partial p}{\partial Q} \\ \frac{\partial q}{\partial \mathbb{P}} & \frac{\partial p}{\partial \mathbb{P}} \end{pmatrix}$$

we have $\begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial \mathbb{P}}{\partial q} & \frac{\partial \mathbb{P}}{\partial p} \end{pmatrix} \begin{pmatrix} \frac{\partial q}{\partial Q} & \frac{\partial q}{\partial \mathbb{P}} \\ \frac{\partial p}{\partial Q} & \frac{\partial p}{\partial \mathbb{P}} \end{pmatrix} = \begin{pmatrix} \frac{\partial Q}{\partial Q} & \frac{\partial Q}{\partial \mathbb{P}} \\ \frac{\partial \mathbb{P}}{\partial Q} & \frac{\partial \mathbb{P}}{\partial \mathbb{P}} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$

$$\Rightarrow M \begin{pmatrix} & I \\ -I & \end{pmatrix} = \begin{pmatrix} & I \\ -I & \end{pmatrix} (M^{-1})^T \Rightarrow \boxed{M \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} M^T = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}}$$

This is called symplectic transformation