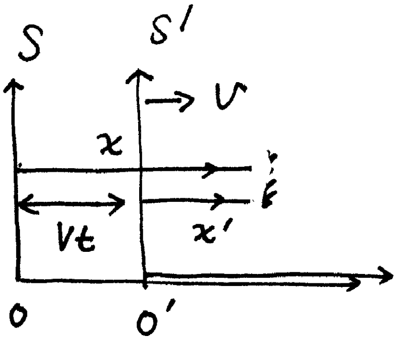
Lect 15 Special Relativity (I)

§ Motivation

Galilean Relativity (invariance): the Newton's laws hold the same form in all the inertial frames. — A moving large boat with closed windows at a constant velocity, the passengers cannot notice its motion as stated by Galileo to defend for Copernicus's sun-centered solar system.



$O' = O$ at $t = 0$

$$\begin{cases} x' = x - Vt \\ y' = y, \quad z' = z, \quad t' = t \end{cases}$$

$$\boxed{\dot{x}' = \dot{x} - V} \leftarrow \text{velocity-addition}$$

$$\ddot{x}' = \ddot{x} \quad \Rightarrow \quad m\vec{a}' = m\vec{a}$$

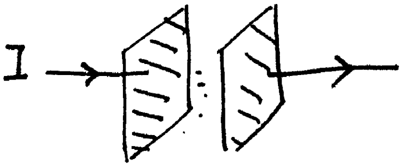
$$\vec{F}' = \vec{F}$$

$\vec{F}' = m\vec{a}'$, in S' $\qquad$ $\vec{F} = m\vec{a}$ in S.

Hower , Maxwell's equation does not obey this Galilean Relativity.

$$\begin{array}{lcl}
\nabla\cdot\vec{E} = \rho/\varepsilon_0 \swarrow \text{Gauss + Coulomb} & & \nabla\cdot\vec{E} = 0\\
\nabla\cdot\vec{B} = 0 & \xrightarrow[\substack{\rho=0\\ \vec{j}=0}]{\text{Vaccum}} & \nabla\cdot\vec{B} = 0\\
\nabla\times\vec{E} = -\frac{\partial}{\partial t}\vec{B} \swarrow \text{Farady} & & \nabla\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}\\
\nabla\times\vec{B} = \underset{\substack{\uparrow\\ \text{Ampere}}}{\mu_0\vec{j}} + \underset{\uparrow}{\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}} \nwarrow \text{Maxwell} & & \nabla\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}
\end{array}$$

displacement current ← a result of Continouity Eq



$$\nabla\cdot(\nabla\times\vec{B}) = \mu_0\left[\nabla\cdot\vec{j} + \varepsilon_0\frac{\partial}{\partial t}\nabla\cdot\vec{E}\right] = 0$$

$$\Leftrightarrow \quad \nabla\cdot\vec{j} + \frac{\partial}{\partial t}\rho = 0$$

$$\vec{j}_{dis} = \frac{\partial\sigma}{\partial t} = \frac{\varepsilon_0\partial\vec{E}}{\partial t} \nwarrow \text{Maxwell's contribution}$$

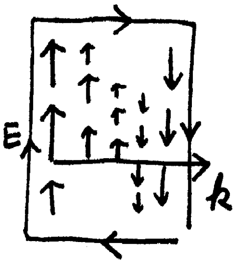
$$\left.\begin{array}{l}
\nabla\times(\nabla\times\vec{E}) = \nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = -\nabla^2\vec{E}\\
\quad\ \ \| \\
\quad -\frac{\partial}{\partial t}\nabla\times\vec{B} = -\mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\vec{E}
\end{array}\right\} \Rightarrow \begin{array}{l}
\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\vec{E}\\
c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3\times10^8 m/s.
\end{array}$$

it looks that Maxell Eq's only valid in a special frame in which $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$. Mechanical waves (eg. water wave) need media. The wave Eq is respect to the frame in which the media is at rest! What's the media of the E-M wave? In 19th's century, it's considered as ether. At that time, people thought Maxwell's equations are only valid in the the frame of ether (vaccum). Einstein did not agree with this!

it meaned in a general frame, Maxwell's equation will be an ugly from.

Einstein felt that Maxwell's equations are so beautiful, it should be valid in any frame! What's needed is NOT to change Maxwell's equation, but to change the Galileo's relativity!

Sine his childhood, Einstein thought over this question: suppose a person travels at the speed of c, what will he see? Then the E-M wave is at rest respect to him and becomes electro-static problem. We know the $\nabla\times\vec{E}\neq 0$ for a E-M wave config.



$\oint \vec{E}\cdot dl \neq 0$

$\nabla\times\vec{E}\neq 0$

This is incompatible with electro-statics.

If we insist Maxwell Eq is correct, the only solution is no-one can travel at the speed of light! Light velocity in any frame is "C".

Michelson-Morley experiment: aim at detect the relative motion of earth respect to ether. by using optics interferometry experiments. but negative results!!! → The relative speed of earth respect to ether is zero??? What's special of earth?

Actually Einstein did not know this experiment before his theory was published!

§ Postulates of special Relativity

① Every inertial frame is equivalent to each other. All the physics laws are the same, we cannot distinguish different inertial frame by looking at the forms of physical laws. — no preferred frame !!

② There's a upper limit for the velocity of signal propagation, which is the "c". finite

→ this upper limit must be the same for all the frames, otherwis
inertial

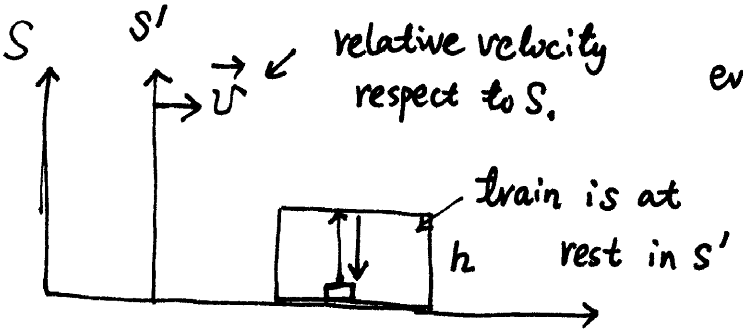
we can distinguish these frames by measuring them.

⟶ light velocity is the same.

§ time dilation — we need to change our understanding of time.

What is time? This is another profound question — time is relate to the fact we become older and dying

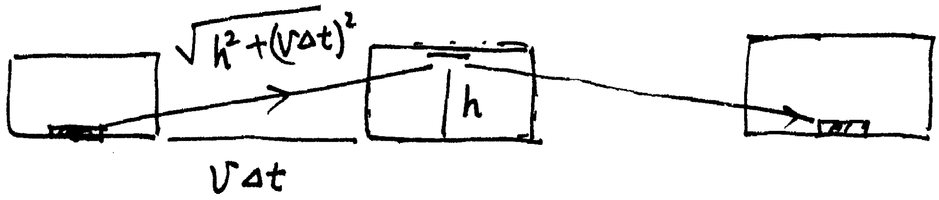
But now we only consider how to measure time.



event "1" a light pulse emitted from the floor

event "2", the light pulse comes back to the floor measured in S', the time interval $\Delta t' = \frac{2h}{c}$.

measured in S



$$\Delta t = 2\sqrt{h^2+(v\Delta t)^2}\,/c \quad \Rightarrow \quad \Delta t = \sqrt{(\Delta t)'^2 + \beta(\Delta t)^2} \qquad \beta = \frac{v}{c}$$

$$(\Delta t)^2 = \frac{(\Delta t')^2}{1-\beta^2} \quad \text{or} \quad \Delta t = \frac{\Delta t'}{\sqrt{1-\beta^2}}$$

$\underline{\Delta t'}$ is measured in the rest frame, which is called "proper" time.

proper time is <u>the shortest</u>! The co-moving clock with train is oberseved run slower.

Example: ① μ-decay τ in the rest frame : $1.5\ \mu s$

from the cosmic ray, say $\gamma = 1.67 = \frac{1}{\sqrt{1-\beta^2}}$

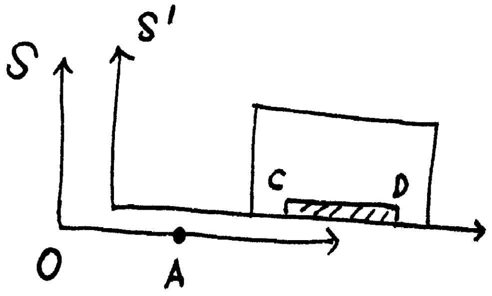
$\beta = 0.8c$

$\Rightarrow \tau = 2.5\ \mu s$.

② atomic clock in the air-plane.

③ GPS

§ Length contraction:



event "1": the end "D" passes "O"

event "2": the end "C" passes "O".

the time interval beween "1" & "2"

in S' frame $\Delta t'$

S frame Δt

$$\Rightarrow \Delta t' = \frac{\Delta t}{\sqrt{1-\beta^2}}$$

(Δt is the proper-time, because events 1, 2 occur at the same locatio

The length of the ruler CD measured in S, $l = v\Delta t$

in S', $l' = v\Delta t'$

$$\Rightarrow \frac{l}{l'} = \frac{\Delta t}{\Delta t'} = \underset{\leq 1.}{\sqrt{1-\beta^2}} \quad : \text{ length contraction.}$$

l' is the length measured in the rest frame — proper length

the longest!

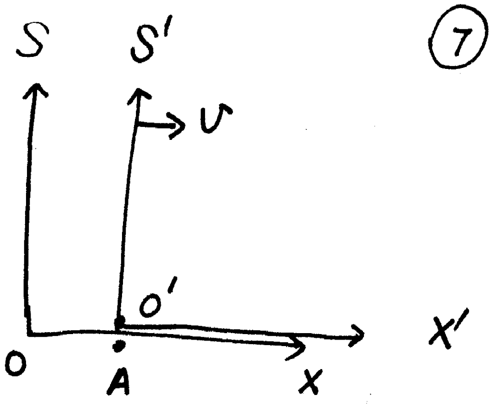
§ Lorentz transformation:

A rotation of space-time ↔ c.f. rotation in 3D space.

$$\begin{pmatrix} \Delta x' \\ c\Delta t' \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix}$$

we need to determine the 4-real coefficients.

Δx, Δt are the spatial & temporal difference between two events measured in S,
$\Delta x'$, $\Delta t'$ S'.



event "1", O' and O coincide

"2", O' and A coincide

$\Rightarrow$ in frame S, $\Delta x = v\Delta t$

S' $\Delta x' = 0$, $\Delta t' = \Delta t\sqrt{1-\beta^2}$

$$\Rightarrow \quad a_1\Delta x + a_2 c\Delta t = 0 \Rightarrow \frac{a_2}{a_1} = \frac{-\Delta x}{c\Delta t} = \frac{-v}{c} = -\beta$$

$\Rightarrow$ $(\Delta t)'$ is the proper time, $\Delta t' = \Delta t\sqrt{1-\beta^2}$

$$c\Delta t' = a_3\Delta x + a_4 c\Delta t = [a_3 v + a_4 c]\Delta t = c\Delta t\sqrt{1-\beta^2}$$

$$\Rightarrow \quad a_3\beta + a_4 = \sqrt{1-\beta^2} \qquad \text{i.e.} \quad \begin{pmatrix} 0 \\ \sqrt{1-\beta^2} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} \beta \\ 1 \end{pmatrix}$$

Let us switch the role of S and S'

respect to S', S is moving along $-\hat{x}$ axis with v,

repeat the above argument, we can have a space-time interval

in S' $\Delta x' = -v\Delta t'$

S $\Delta x = 0$, $\Delta t = \Delta t'\sqrt{1-\beta^2}$

$$\boxed{\begin{matrix} \Delta x \\ \Delta t \end{matrix} \leftrightarrow \begin{matrix} \Delta x' \\ \Delta t' \end{matrix} \qquad v \leftrightarrow -v}$$

$$\begin{pmatrix} -v\Delta t' \\ c\Delta t' \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \underset{c\Delta t'}{\begin{pmatrix} 0 \\ \sqrt{1-\beta^2} \end{pmatrix}} \Rightarrow \begin{pmatrix} -\beta \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{1-\beta^2} \end{pmatrix}$$

$$\Rightarrow \quad a_2 = \frac{-\beta}{\sqrt{1-\beta^2}} \qquad a_4 = \frac{1}{\sqrt{1-\beta^2}} \qquad \Rightarrow \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix}$$

$$a_1 = \frac{1}{\sqrt{1-\beta^2}} \qquad a_3 = \frac{-\beta}{\sqrt{1-\beta^2}}$$

or $$\begin{pmatrix} \Delta x' \\ c\Delta t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix} \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

Important result: ① the length of space-time interval

$(\Delta S)^2 = (\Delta x)^2 - (c\Delta t)^2$ is invariant under Lorentz transform

$$(\Delta S)^2 = (\Delta x, c\Delta t) \underbrace{\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}}_{\text{metric}} \begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix}$$

$$u = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix}$$

$$= [\Delta x', c\Delta t'] \, u^T \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} u \begin{pmatrix} \Delta x' \\ (c\Delta t)' \end{pmatrix}$$

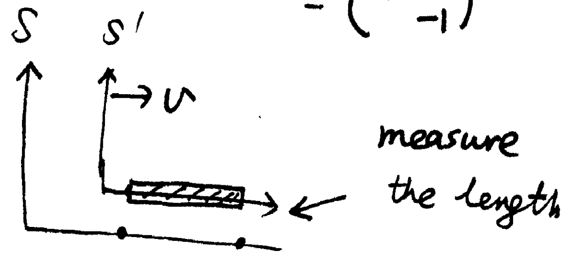
$$u^T = u$$

$$= [\Delta x', c\Delta t'] \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} \Delta x' \\ c\Delta t' \end{bmatrix} = (\Delta S')^2$$

$$u^T \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} u = \gamma^2 \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ \beta & -1 \end{pmatrix} = \gamma^2 \begin{pmatrix} \gamma^{-2} & \\ & -\gamma^{-2} \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

② proper length

measure length of a moving ruler



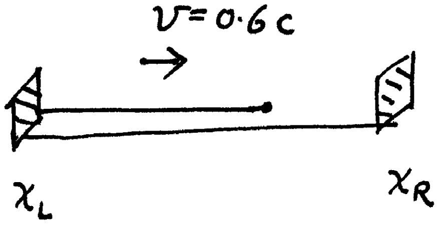
in S $\quad \Delta t = 0, \ \Delta x = \ell \quad$ (we need $\Delta t = 0$ becuse ruler is moving

in S' $\quad \Delta t' = -\gamma\beta\frac{\ell}{c} \quad \underset{\substack{\uparrow \\ \text{proper}}}{\Delta x'} = \gamma\ell \quad$ (the ruler is static, so $\Delta t' \neq 0$ is fine) $\Rightarrow \boxed{\Delta x' / \Delta x = \gamma}$

③ relativity of simultaneity

if $\Delta t = 0$, $\Rightarrow \Delta t' = -\gamma\beta\Delta x \neq 0$.

the snake-paradox: P613 a snake with proper length 100 cm, travel respect to the table $v = 0.6c$. Two cleavers with distance 100cm cut the same time in the lab frame. (S).



In the frame (S), $\Delta t = 0$; $t_L = t_R$

$\Delta x = 100$ cm ; $x_R - x_L = 100$ cm

length of snake $\frac{100\,\text{cm}}{\gamma} = \sqrt{1-0.6^2}\cdot 100 = 80$cm. < 100 cm

no hurm!

In the snake frame: the snake felt the length between two cleavers $\frac{100}{\gamma} = 80$ cm. Will it be cut?

< 100 cm

$$\begin{pmatrix} \Delta x' = x'_R - x'_L \\ c\Delta t' = \underset{c}{(}t'_R - t'_L) \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} 100\,\text{cm} \\ 0 \end{pmatrix}$$

$\gamma = \frac{1}{0.8} = 1.25$

$\beta = 0.6$

$\uparrow$

$$\begin{pmatrix} 1.25 & -0.75 \\ -0.75 & 1.25 \end{pmatrix}$$

$$\begin{cases} \Delta x' = x'_R - x'_L = 125\,\text{cm} \\ c\Delta t' = -75\,\text{cm} \end{cases}$$

the right cleave fell at $\frac{-75}{3\times10^8}$ s, at 125cm.

No problem.

④ What's relative,/ absolute?

$(\Delta S)^2$ does not change. $(\Delta s)^2 = x^2 - c^2 \Delta t^2$

$(\Delta s)^2 > 0$: space-like

$(\Delta s)^2 = 0$ light like

$(\Delta s)^2 < 0$ time-like.

if $(\Delta s)^2 \leq 0$, it means the two events can built up causalitly relation.

$$|\Delta t| \geqslant \left|\frac{\Delta x}{c}\right|$$

then $c\Delta t' = \gamma(-\beta \Delta x + c\Delta t) = \gamma c\left(\Delta t - \frac{\beta \Delta x}{c}\right)$

$|\Delta t| > \left|\frac{\Delta x}{c}\right| > \frac{\beta}{c}|\Delta x|$ ⇒ Δt and $\Delta t'$ has the same sign.

the sequence of two light-like time-like events cannot be reversed!!
(with causality relation)

but for two space-like events (without causality relation)

their sequence does can be reversed!