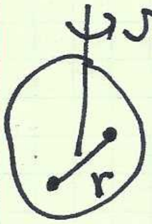
Lect 4 Rigid body (I) fixed axis rotation

§ What's rigid body?

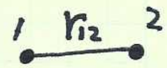
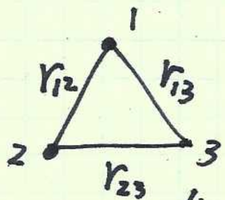
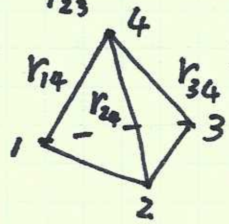
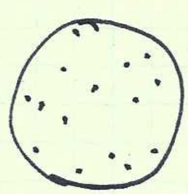


The distance r between any two points is a constant during motion, i.e. no shape deformation.

No elastisity!

how many continous degees of freedom: 3 translational, 3 rotational $\}$ = 6.

we can check

N	# degrees of freedom	
1	3	
2	$2 \times 3 - 1 = 5$	
3	$3 \times 3 - 3 = 6$	
4	no extra continous degrees of freedom, remain 6. (please prove).	
5	6	
⋮	⋮	
N	6	

§ center of mass: (CM) ← not necessary rigid body

A collection of particles $\alpha = 1, 2, \cdots N$ with masses m_α, and location r_α. Define their center of mass $\boxed{\vec{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha}$,

where $M = \sum_{\alpha=1}^{N} m_\alpha$.

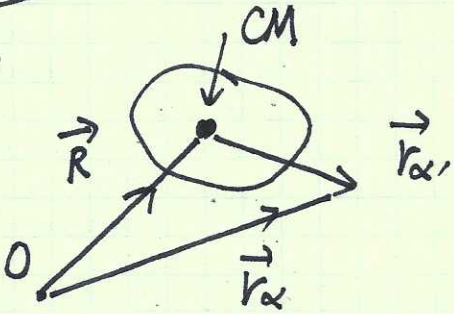
- then total linear momentum $\boxed{\vec{P}_{tot} = \sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha = M\dot{\vec{R}}}$

and $\boxed{\vec{F}^{ext} = M\ddot{\vec{R}}}$ (internal forces cancel due to Newton's third law).

↓ not true for charged particles.

- total angular momentum

$$\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha' \quad \text{— coordinate respect to the CM}$$



$$\vec{l}_\alpha = \vec{r}_\alpha \times \vec{P}_\alpha = \vec{R}\times\vec{P}_\alpha + \vec{r}_\alpha' \times \vec{P}_\alpha$$

$$\vec{L}_{tot} = \sum_{\alpha=1}^{N} \vec{l}_\alpha = \vec{R}\times\sum_{\alpha=1}^{N}\vec{P}_\alpha + \sum_{\alpha=1}^{N}\vec{r}_\alpha' \times m_\alpha \dot{\vec{r}}_\alpha$$

$$= \vec{R}\times\vec{P}_{tot} + \sum_{\alpha=1}^{N}\vec{r}_\alpha' \times m_\alpha(\dot{\vec{r}}_\alpha' + \dot{\vec{R}})$$

$$= \underbrace{\vec{R}\times\vec{P}_{tot}}_{\vec{L}_{orbit}} + \underbrace{\sum_{\alpha=1}^{N}\vec{r}_\alpha' \times m_\alpha \dot{\vec{r}}_\alpha'}_{\text{internal angular momentum with respect to CM } \vec{L}_{spin}} + \underbrace{\left(\sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha'\right)}_{=0} \times \dot{\vec{R}}$$

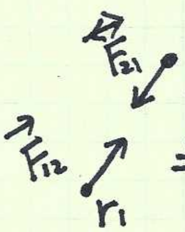
(please prove it!)

$$\dot{\vec{L}}_{orbit} = \underbrace{\dot{\vec{R}} \times \vec{P}}_{0} + \vec{R} \times \dot{\vec{P}} = \vec{R} \times \vec{F}^{ext}$$

$$\dot{\vec{L}}_{tot} = \underbrace{\sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{F}_\alpha^{ext}}_{} = \sum_{\alpha=1}^{N} (\vec{r}_\alpha' + \vec{R}) \times \vec{F}_\alpha^{ex}$$

$$= \dot{\vec{L}}_{orbit} + \sum_{\alpha=1}^{N} \vec{r}_\alpha' \times \vec{F}_\alpha^{ext}$$

contributions from internal forces vanishe due to Newton 3rd law. (why?)



$\Rightarrow$ total torque zero for internal forces.

$$\Rightarrow \boxed{\dot{\vec{L}}_{spin} = \sum_{\alpha=1}^{N} \vec{r}_\alpha' \times \vec{F}_\alpha^{ext}}$$

torque respect to CM.

QM: electron also has orbital angular momentum + spin.

angular momentum quata $\frac{\hbar}{2}$.

§ Kinetic energy (KE)

$$T = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2} = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha (\dot{\vec{R}}^2 + 2\dot{\vec{R}} \cdot \dot{\vec{r}}_\alpha' + \dot{\vec{r}}_\alpha'^{\,2})$$

$$= \frac{1}{2}\left(\sum_{\alpha=1}^{N} m_\alpha\right) \dot{\vec{R}}^2 + \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha'^{\,2} + \underbrace{\left(\sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha'\right)}_{0,\text{ please prove}} \cdot \dot{\vec{R}}$$

$$\Rightarrow \boxed{T = \frac{1}{2} M \dot{R}^2 + \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha \dot{r}_\alpha'^{\,2}}$$

center of mass KE

internal KE

This decomposition is also valid when $\vec{R}$ is not the center of mass if $\dot{\vec{R}} = 0$ (even instantly)

$$\dot{\vec{R}} = 0 \Rightarrow T = \frac{1}{2} \sum_{\alpha=1}^{N} m \dot{r}_\alpha^2,$$

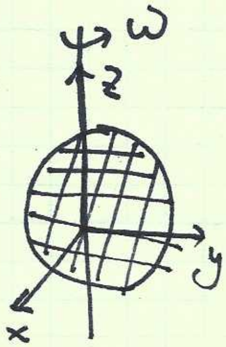
useful for describing rolling, and fixed point rotation.

§ potential energy

$$\underbrace{U}_{\uparrow} = U^{ext} + U^{int} \leftarrow \text{const for a rigid body}$$

sum of all potential energy of every particle

§ laws of rotation around a fixed axis



$$\vec{L} = \sum_{\alpha=1}^{N} \vec{l}_\alpha = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times m_\alpha \vec{v}_\alpha$$

$$\vec{v}_\alpha = \vec{\omega} \times \vec{r}_\alpha \quad \text{where } \vec{\omega} = (0, 0, \omega)$$

$$\vec{r}_\alpha = (x_\alpha, y_\alpha, z_\alpha)$$

$$\Rightarrow \vec{r}_\alpha \times \vec{v}_\alpha = \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha) = \vec{\omega}\, r_\alpha^2 - \vec{r}_\alpha (\vec{\omega} \cdot \vec{r}_\alpha)$$

$$= (0, 0, \omega(x_\alpha^2 + y_\alpha^2 + z_\alpha^2)) - \omega z_\alpha (x_\alpha, y_\alpha, z_\alpha)$$

$$= \omega(-z_\alpha x_\alpha, -z_\alpha y_\alpha, x_\alpha^2 + y_\alpha^2)$$

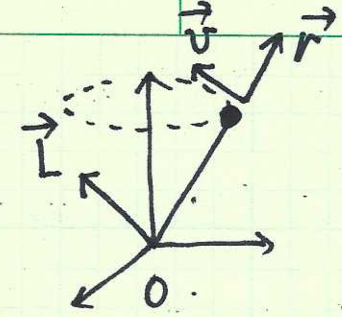
$$L_z = \sum l_{\alpha,z} = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2)\, \omega = I_z\, \omega$$

where $I_z = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2) \longrightarrow \int dx\,dy\,dz\ \rho(x^2 + y^2)$ for continuum

$$L_x = -\left(\sum m_\alpha x_\alpha z_\alpha\right) \omega \neq 0,\ L_y = -\left(\sum m_\alpha y_\alpha z_\alpha\right) \omega \neq 0.$$

generally speaking. $\vec{L} \not\parallel \vec{\omega}$

a simple example



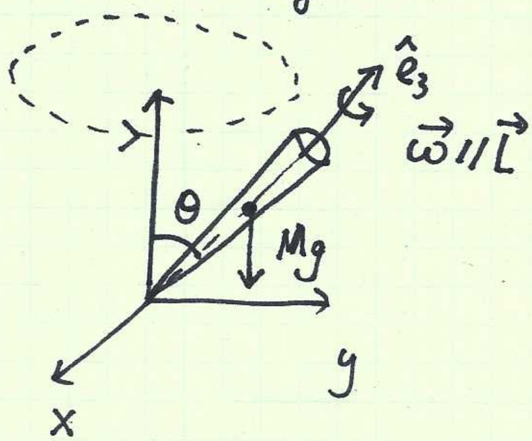
§ Kinetic energy

$$T = \frac{1}{2}\sum_\alpha m_\alpha v_\alpha^2 = \frac{1}{2}\sum_\alpha m_\alpha \omega^2 (x_\alpha^2 + y_\alpha^2) = \boxed{\frac{1}{2} I_z \omega^2 = T}$$

$$\vec{v}_\alpha = \vec{\omega} \times \vec{r}_\alpha = -\omega(y_\alpha, x_\alpha, 0)$$

§ Example: precession of a top due to a week torque

the top is symmetric around its axis $\hat{e}_3$, and its spinning fast around $\hat{e}_3$. And in this case

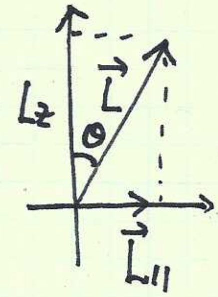


$\vec{L} \approx I_3 \vec{\omega}$ if we negelect the slow precession described below.

The gravity has a torque $\vec{R} \times M\vec{g} \approx \frac{d\vec{L}}{dt} = \frac{d\vec{L}_{\parallel}}{dt}$

$\Rightarrow R M g \sin\theta = I_3 \omega \sin\theta \, \Omega$

Ω is along $\hat{z}$, $d\vec{L}$ is into the page



$$\Rightarrow \boxed{\vec{\Omega} = \frac{RM}{I_3 \omega} g \, \hat{z}}$$