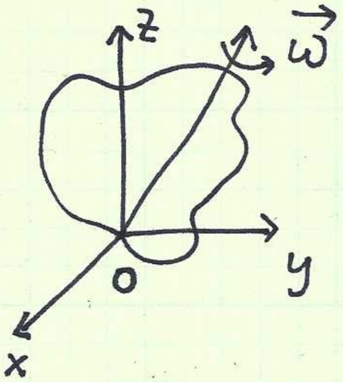
Lecture 5 Inertial tensor, Principle axis

Fixed point rotation



§ definition: moment of inertial

Rotation around any axis

$$\vec{\omega} = (\omega_x, \omega_y, \omega_z)$$

$$\vec{L} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha = \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$$

$$\vec{r} \times (\vec{\omega} \times \vec{r}) = \vec{\omega} r^2 - \vec{r}(\vec{\omega} \cdot \vec{r}) = (\omega_x r^2, \omega_y r^2, \omega_z r^2) - (\vec{\omega} \cdot \vec{r})(x, y, z)$$

$$= [(y^2+z^2)\omega_x - xy\,\omega_y - xz\,\omega_z,$$
$$-yx\,\omega_x + (z^2+x^2)\omega_y - yz\,\omega_z,$$
$$-zx\,\omega_x - zy\,\omega_y + (x^2+y^2)\omega_z]$$

$$\Rightarrow \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

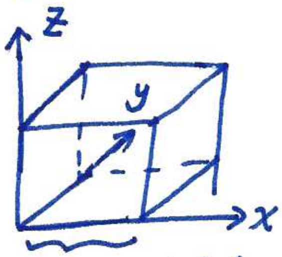
$$\Rightarrow L_a = I_{ab}\,\omega_b$$

where $I_{ab} = \delta_{ab}\left(\sum_{\alpha=1}^{N} m_\alpha r_\alpha^2\right) - \sum_{\alpha=1}^{N} m_\alpha x_a x_b$

$$= \int dx\,dy\,dz\ \rho\,[r^2 \delta_{ab} - x_a x_b]$$ ← $I_{ab} = I_{ba}$ symmetric tensor

Example. inertia tensor of a solid cube, rotating

① around the corner



a, total mass M

$$I_{xx} = \int_0^a dx \int_0^a dy \int_0^a dz\ \rho (y^2+z^2) = \rho a^2 \left[\frac{a^3}{3}\right] \times 2 = \frac{2a^2}{3} M$$

$$I_{yy} = I_{zz} = I_{xx}$$

$$I_{xy} = -\int_0^a dx \int_0^a dy \int_0^a dz\ \rho\, xy = -\rho\, \frac{a^2}{2} \frac{a^2}{2} \cdot a = -\frac{a^2}{4} M$$

$$\Rightarrow I = Ma^2 \begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix}$$

If the cube is rotate around z-axis, $\vec{\omega} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \Rightarrow$

$$\vec{L} = Ma^2\omega \begin{bmatrix} -1/4 \\ -1/4 \\ \frac{2}{3} \end{bmatrix}.$$

② if the cube rotates around it center

$$I_{xx} = \int_{-a/2}^{a/2} dx\, dy\, dz\ \rho (y^2+z^2) = \rho a^2 \frac{2}{3} \left(\frac{a}{2}\right)^3 \times 2 = \frac{a^2}{6} M$$

$$I_{yy} = I_{zz} = I_{xx}$$

$$I_{xy} = I_{yz} = I_{zx} = 0$$

$$\Rightarrow I = \frac{Ma^2}{6} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \Rightarrow \vec{L} = \frac{Ma^2}{6} \vec{\omega}$$

"I" depends on the location of the origin & the orientation of the axis.

§ Kinetic energy of fix-point rotation

$$T = \frac{1}{2}\sum_\alpha m_\alpha \vec{v}_\alpha^{\,2}$$

$$\vec{v} = \vec{\omega}\times\vec{r} \quad\Rightarrow\quad v^2 = \omega^2 r^2 - (\vec{\omega}\cdot\vec{r})^2$$

$$= (\omega_x^2+\omega_y^2+\omega_z^2)(x^2+y^2+z^2) - (\omega_x x + \omega_y y + \omega_z z)^2$$

$$= \omega_x^2(y^2+z^2) + \omega_y^2(z^2+x^2) + \omega_z^2(x^2+y^2)$$

$$- 2\omega_x\omega_y xy - 2\omega_y\omega_z yz - 2\omega_z\omega_x zx$$

$$\Rightarrow T = \frac{1}{2}\sum_\alpha m_\alpha \vec{v}_\alpha^{\,2}$$

$$= \frac{1}{2}[\omega_x\ \omega_y\ \omega_z]\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}\end{bmatrix}\begin{bmatrix}\omega_x \\ \omega_y \\ \omega_z\end{bmatrix}$$

$$\Rightarrow T = \frac{1}{2}\omega_a I_{ab}\omega_b = \frac{1}{2}\vec{\omega}\cdot\vec{L}$$

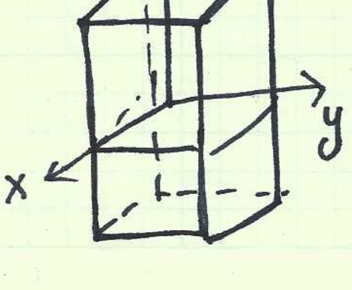
I_{ab} is a 3×3 symmetric real matrix, which can be diagonalized by an orthogonal matrix. The diagonalized form of $I_{ab} \sim (\lambda_1, \lambda_2, \lambda_3)$

$$\Rightarrow T = \frac{1}{2}(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2)$$

in the frame of principle axis.

$(\omega_1, \omega_2, \omega_3)$ is the projection of $\vec{\omega}$ along principle axis

Ex: Cuboids



$$I_{xy} = I_{yz} = I_{zx} = 0$$

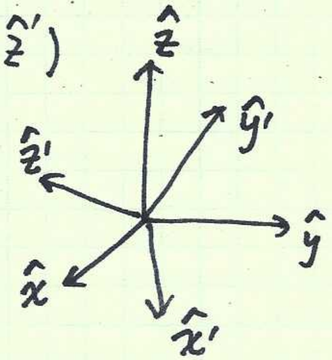
$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz}\end{bmatrix}$$

xyz are principle axis.

§ The transformation of the inertial tensor

$$I_{ab} = \int dx\,dy\,dz \;\; \rho\left(r^2 \delta_{ab} - x_a x_b\right)$$

Generally speaking, I_{ab} is non-diagonal matrix. We need to find a convenient frame in which I_{ab} is diagonal. Consider we change our frame basis, from $(\hat{x}, \hat{y}, \hat{z}) \to (\hat{x}', \hat{y}', \hat{z}')$



$$(\hat{x}, \hat{y}, \hat{z}) \cdot \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = (\hat{x}', \hat{y}', \hat{z}')$$

or

$$\begin{cases} \hat{x}' = T_{11}\hat{x} + T_{21}\hat{y} + T_{31}\hat{z} \\ \hat{y}' = T_{12}\hat{x} + T_{22}\hat{y} + T_{32}\hat{z} \\ \hat{z}' = T_{13}\hat{x} + T_{23}\hat{y} + T_{33}\hat{z} \end{cases}$$

T is an orthogonal matrix and thus $\hat{x}', \hat{y}', \hat{z}'$ is also a orth-normal basis.

The coordinate transforms as

$$\vec{r} = (\hat{x}\;\hat{y}\;\hat{z}) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (\hat{x}', \hat{y}', z') \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = (\hat{x}\;\hat{y}\;\hat{z})\, T \begin{pmatrix} x' \\ y' \\ y' \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = T \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}. \quad \text{or} \quad x_a = T_{ab}\, x'_b$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = T^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = T^{t} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \boxed{\begin{aligned} x'_a &= (T^t)_{ab}\, x_b \\ &= T_{ba}\, x_b \end{aligned}}$$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is called vector because it follows this transition.

Then how does the moment of inertial transform?

$$I'_{ab} = \int dx'dy'dz' \; [r'^2 \delta_{ab} - x'_a x'_b] \rho(x' y' z')$$

$$\int dx'dy'dz' = \int dx\,dy\,dz, \quad r'^2 = r^2, \quad \rho(x',y',z') = \rho(x,y,z)$$

$$x'_a x'_b = (T^t)_{aa'} x_{a'} (T^t)_{bb'} x_{b'} = (T^t)_{aa'} x_{a'} x_{b'} (T)_{b'b}$$

$$\Rightarrow I'_{ab} = \int dxdydz \, \rho(xyz) \, [r^2 (T^t)_{aa'} \delta_{a'b'} (T)_{b'b} - (T^t)_{aa'} x_{a'} x_{b'} (T)_{b'b}]$$

$$= (T^t)_{aa'} \left[\int dxdydz \, \rho(r) (\underbrace{r^2}_{\delta_{a'b'}} - x_{a'} x_{b'}) \right] (T)_{b'b}$$

$$= (T^t)_{aa'} I_{a'b'} (T)_{b'b}$$

or $\boxed{I' = T^{-1} I \, T}$

We know that I, a symmetric matrix, which can be diagonalized by orthogonal matrix. We can choose suitable T to make

$I' = \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3]$, with corresponding orientation

principle axis

of

$$(\hat{x}', \hat{y}', \hat{z}') = (\hat{x}, \hat{y}, \hat{z}) \, T$$

or $\boxed{\hat{X}'_a = \hat{X}_b T_{ba}}$

i.e. for $\hat{x}'$ choose the first column of T, and so on.

If $\vec{\omega}$ is along the principle axis, then $\vec{L} \parallel \vec{\omega}$. In other word

$$L_a = I_{ab} \omega_b = \lambda \omega_a ,$$ thus $\vec{\omega}$ is eigen vector of I.

$$\Rightarrow \quad \Leftrightarrow \quad \underline{\det [I_{ab} - \lambda \delta_{ab}] = 0} . \quad (*)$$

From (*), the characteric equation, we can find the eigenvalues and the corresponding eigenvectors of $\vec{\omega}$, which are along the principle axes.

Example: The example in last lecture, a uniform cube rotating around its corner

$$I = \mu \begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix} \qquad \mu = \frac{Ma^2}{12}$$

$$\det[I_{ab} - \lambda \delta_{ab}] = \left| \begin{bmatrix} 8\mu - \lambda & -3\mu & -3\mu \\ -3\mu & 8\mu - \lambda & -3\mu \\ -3\mu & -3\mu & 8\mu - \lambda \end{bmatrix} \right| = (2\mu - \lambda)(11\mu - \lambda)^2 = 0$$

$\Rightarrow \quad \lambda_1 = 2\mu , \qquad \lambda_2 = \lambda_3 = 11\mu .$ (eigenvalues found!)

For principle axis 1

$$[I_{ab} - \lambda_1 \delta_{ab}] \omega_b = 0 \Rightarrow \begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = 0$$

$\Rightarrow \quad \omega_x = \omega_y = \omega_z \Rightarrow$ the principle axis $\vec{e}_1 = \frac{1}{\sqrt{3}} (1, 1, 1)$

For principle axes 2, & 3 $\Rightarrow$

$$(I_{ab} - \lambda_{2,3}\,\delta_{ab})\,\omega_b = 0 \Rightarrow (-3\mu)\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = 0$$

$$\Rightarrow \omega_x + \omega_y + \omega_z = 0.$$

We have a freedom to choose $\hat{e}_2$ & $\hat{e}_3$

say $\hat{e}_2 = \frac{1}{\sqrt{2}}(1,-1,0)$, $\hat{e}_3 = \frac{1}{\sqrt{6}}(1,1,-2)$,

$\hat{e}_1, \hat{e}_2, \hat{e}_3$ are perpendicular to each other.