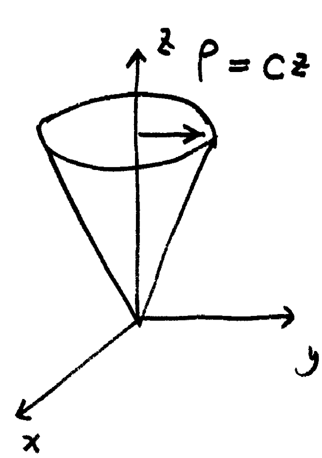
Lect 9 Hamiltonian mechanics (II)

More example: a mass on a cone.



set z and $\dot{\phi}$ as independent variables

$$T = \frac{1}{2} m \left(\dot{\rho}^2 + (\rho\dot{\phi})^2 + \dot{z}^2 \right)$$

$$\rho = cz$$

$$\Rightarrow T = \frac{1}{2} m \left[(c^2+1)\dot{z}^2 + c^2 z^2 \dot{\phi}^2 \right]$$

$$\cancel{H = T + V = \frac{1}{2}} \qquad P_z = \frac{\partial T}{\partial \dot{z}} = m(c^2+1)\dot{z}, \qquad P_\phi = \frac{\partial T}{\partial \dot{\phi}} = m c^2 z^2 \dot{\phi}$$

$$\Rightarrow H = T + V = \frac{1}{2m}\left[\frac{P_z^2}{(c^2+1)} + \frac{P_\phi^2}{c^2 z^2} \right] + mgz$$

$$\boxed{\begin{aligned} &\dot{z} = \frac{\partial H}{\partial P_z} = \frac{P_z}{m(c^2+1)} && \dot{\phi} = \frac{\partial H}{\partial P_\phi} = \frac{P_\phi}{m c^2 z^2} \\ &\dot{P}_z = -\frac{\partial H}{\partial z} = \frac{P_\phi^2}{m c^2 z^3} - mg && \dot{P}_\phi = -\frac{\partial H}{\partial \phi} = 0 \quad \text{i.e. } L_z \text{ conserved.} \end{aligned}}$$

Let consider a fixed z solution $\Rightarrow P_z = 0$

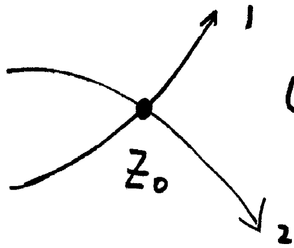
$$\frac{P_\phi^2}{m c^2 z^3} - mg = 0 \Rightarrow P_\phi = \pm\sqrt{m^2 c^2 g z^3}.$$

For more detailed solution Prob 13.14, 13.17.

§ Phase space-orbits (q, p) $\dot{q} = +\frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial q}$

initial point $Z_0 = (q_0, p_0)$ uniquely determin its trajectory as time evolves.

★ There's no two different phase-space orbits can cross one another.



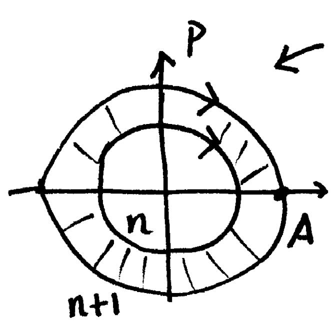
(not allowed! Even trajectory 1 and 2 cross at Z_0 at different time!)

otherwise, 1 & 2 have to be the same orbit.

Example: 1d harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 = E = m\omega^2 A^2$$

$$\left.\begin{aligned} x &= A\cos(\omega t + \varphi) \\ p &= -m\omega A \sin(\omega t + \varphi) \end{aligned}\right\} \Rightarrow \left(\frac{x}{A}\right)^2 + \left(\frac{p}{m\omega A}\right)^2 = 1$$



← Classical picture

QM: E is quantized

$$E = m\omega^2 A^2 = (n + \tfrac{1}{2})\hbar\omega$$

$$A = \sqrt{\frac{(n+\frac{1}{2})\hbar}{m\omega}}$$

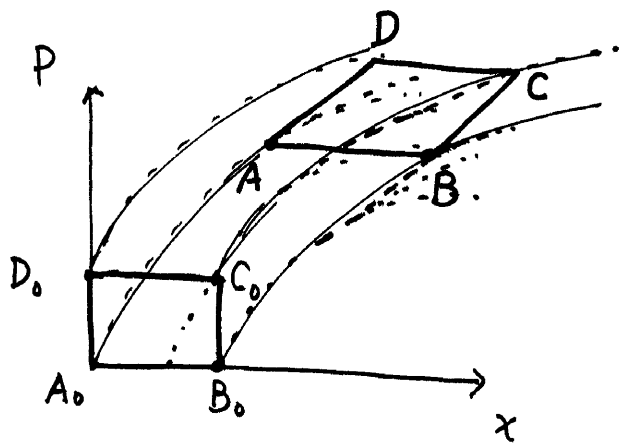
Area between the n+1 and n th orbit is $\underline{\underline{h}}$.

$$\oint p \cdot dq = (n + \tfrac{1}{2}) h$$

Example: falling - body

$$H = \frac{p^2}{2m} - mgx \quad \Rightarrow \quad \boxed{\dot{x} = \frac{p}{m}, \quad \dot{p} = mg}$$

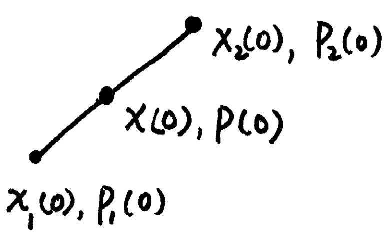
$$\left.\begin{aligned} p &= p_0 + mgt \\ x &= x_0 + \frac{p_0}{m}t + \frac{1}{2}gt^2 \end{aligned}\right\} \Rightarrow \quad x - x_0 = \frac{p^2 - p_0^2}{2m^2 g}$$



★ initial states inside the rectangle $A_0(0,0)$, $B_0(X,0)$, $C_0(0,P)$, $D_0(X,P)$.

Q: what's the time-evolution of this rectangle?

First let me prove a straight line, is still a straight line



$$X(0) = \lambda X_1(0) + (1-\lambda) X_2(0)$$

$$P(0) = \lambda P_1(0) + (1-\lambda) P_2(0)$$

$$\Rightarrow \quad P(t) = P(0) + mgt$$

$$= \lambda P_1(0) + (1-\lambda) P_2(0) + \lambda mgt + (1-\lambda) mgt$$

$$= \lambda (P_1(0) + mgt) + (1-\lambda)(P_2(0) + mgt) = \lambda P_1(t) + (1-\lambda) P_2(t)$$

$$X(t) = X(0) + \frac{P_0}{m}t + \frac{1}{2}gt^2$$

$$= \lambda \left[X_1(0) + \frac{P_1(0)}{m}t + \frac{1}{2}gt^2 \right] + [1-\lambda]\left[X_2(0) + \frac{P_2(0)}{m}t + \frac{1}{2}gt^2 \right] = \lambda X_1(t) + (1-\lambda) X_2(t)$$

Thus A_0B_0, B_0C_0, C_0D_0, D_0A_0 evolve into lines of AB, BC, CD, DA.

it's easy to show $\left.\begin{array}{l} AB \\ CD \end{array}\right\}$ remains horizontal, since their initial momenta are the same.

the length $\left.\begin{array}{l} AB = A_0B_0 \\ CD = C_0D_0 \end{array}\right\}$ because theire inital velocity is the same.

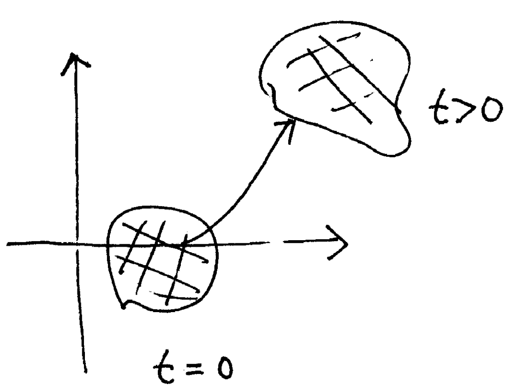
The height between AB and CD is invariant, because they experience the acceleration for the same time interval.

$\Rightarrow$ Area of $ABCD$ is invariant with time.

§ Liouville's theorem

A collection of particles with initial condition $z_a(0) = (q_i^a(0), p_i^a(0))$

$a = 1, \cdots N$ (index of particle), i: index of coordinates

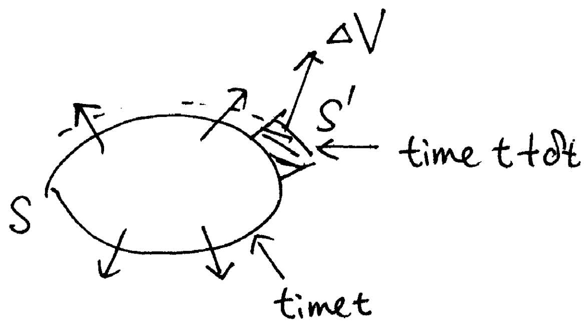


phase space velocity

$$\dot{z}_a = (\dot{q}^a, \dot{p}^a) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q} \right)$$

☆ The volume in the phase space does not charge.

Let us consider at time t, the volume ~~[illegible]~~ enclosed by a surface S. How the surface changes with time?



$$\Delta V = \hat{n} \cdot \vec{v}\, dA$$

$$\delta V = \oiint \vec{v} \cdot d\vec{A} = \iiint_V dp_i\, dq_i\, (\nabla \cdot \vec{v})$$

phase space

$$\vec{v} = (\dot{p}, \dot{q})$$

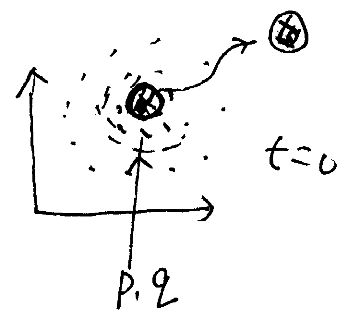
$$\nabla_{p_i} \dot{p}_i + \nabla_{q_i} \dot{q}_i = \frac{\partial}{\partial p_i}\left(\frac{-\partial H}{\partial q_i}\right) + \frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) = 0.$$

The flow of (p_i, q_i) in the phase space is divergence free.

$\Rightarrow \delta V = 0.$

~~The density~~

Essemble: intial states probility: $\rho_0(p, q)$



as time evolves

$$\frac{d}{dt} \rho(p(t), q(t)) = 0$$