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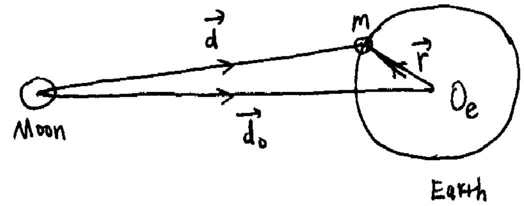
Problem 9.6.

Condition for surface of ocean in equilibrium:

Since water cannot provide shearing force, such a condition is that the force exerted on a drop of water at the surface by other parts of ocean water should be perpendicular to the surface.

Work in a non-spinning frame with origin fixed at the centre of earth.



$$\vec{F}_{tide} = \vec{r}\cdot\nabla\left(-\frac{GM_{moon}m}{R^2}\hat{R}\right)\Big|_{\vec{R}=\vec{d_0}} \quad (\text{to lowest order approximation})$$

$$= -GM_{moon}m \ \vec{r}\cdot\nabla\left(\frac{\hat{R}}{R^2}\right)\Big|_{\vec{R}=\vec{d_0}}$$

in which

$$\nabla\left(\frac{\hat{R}}{R^2}\right) = \nabla\left(\frac{\vec{R}}{R^3}\right)$$

$$= (\partial_x\hat{i} + \partial_y\hat{j} + \partial_z\hat{k})\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{R^3}\right)$$

$$= \nabla\left(\frac{1}{R^3}\right)\vec{R} + \frac{1}{R^3}\nabla(\vec{R}) \quad (\text{Notice that } \nabla \text{ should be put forward})$$

$$= -\frac{3}{R^4}(\nabla R)\vec{R} + \frac{1}{R^3}\nabla\vec{R}$$

$$= -\frac{3}{R^4}\frac{\vec{R}}{R}\vec{R} + \frac{1}{R^3}\overleftrightarrow{I}$$

$$= -\frac{3}{R^3}\hat{R}\hat{R} + \frac{1}{R^3}\overleftrightarrow{I} = \frac{1}{R^3}(\overleftrightarrow{I} - 3\hat{R}\hat{R})$$

since

$$\nabla R = (\partial_x\hat{i} + \partial_y\hat{j} + \partial_z\hat{k})R = \frac{1}{R}\vec{R}$$

$$\nabla\vec{R} = (\partial_x\hat{i} + \partial_y\hat{j} + \partial_z\hat{k})(x\hat{i}+y\hat{j}+z\hat{k}) = \hat{i}\hat{i} + \hat{j}\hat{j} + \hat{k}\hat{k} \ (= \overleftrightarrow{I})$$

Hence

$$\vec{F}_{tide} = -GM_{moon}m\,\vec{r}\cdot\frac{1}{d_0^3}(\overleftrightarrow{I} - 3\hat{d_0}\hat{d_0})$$

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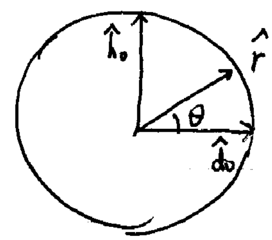
$$= \frac{GM_{moon}m}{d_0^2}\frac{r}{d_0}\left(3(\hat{d}_0\cdot\hat{r})\hat{d}_0 - \hat{r}\right)$$



Plugging in

$$\hat{r} = \hat{d}_0\cos\theta + \hat{h}_0\sin\theta$$
$$\hat{d}_0\cdot\hat{r} = \cos\theta$$

We have

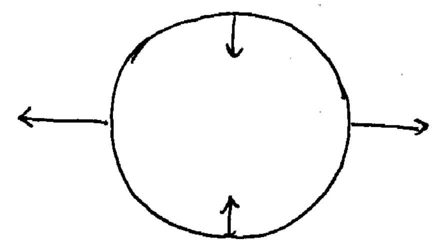
$$\vec{F}_{tide} = \frac{GM_{moon}m}{d_0^2}\frac{r}{d_0}\left(2\hat{d}_0\cos\theta - \hat{h}_0\sin\theta\right)$$

$$\Rightarrow \quad \vec{F}_{tide}(\theta=0) = \frac{GM_{moon}}{d_0^2}\frac{r}{d_0}\,2\hat{d}_0$$

$$\vec{F}_{tide}(\theta=\pi) = \frac{GM_{moon}}{d_0^2}\frac{r}{d_0}(-2\hat{d}_0)$$

$$\vec{F}_{tide}(\theta=\tfrac{\pi}{2}) = \frac{GM_{moon}m}{d_0^2}\frac{r}{d_0}(-\hat{h}_0)$$

$$\vec{F}_{tide}(\theta=-\tfrac{\pi}{2}) = \frac{GM_{moon}m}{d_0^2}\frac{r}{d_0}(\hat{h}_0)$$



Now let's try to write $\vec{F}_{tide}$ in terms of gradient of a potential.

$$\vec{F}_{tide} = \vec{r}\cdot(-GM_{moon}m)\,\nabla\left(\frac{\hat{R}}{R^2}\right)\Big|_{\vec{R}=\vec{d}_0}$$

where

$$-(GM_{moon}m)\,\nabla\left(\frac{\hat{R}}{R^2}\right)\Big|_{\vec{R}=\vec{d}_0} \overset{\text{denoted}}{\underset{\text{by}}{=}} \overset{\rightarrow\rightarrow}{A}$$

is a constant, so

$$\vec{F}_{tide} = \vec{r}\cdot\overset{\rightarrow\rightarrow}{A}.$$

$\overset{\rightarrow\rightarrow}{A}$ is symmetric, so that $\vec{r}\cdot\overset{\rightarrow\rightarrow}{A} = \overset{\rightarrow\rightarrow}{A}\cdot\vec{r}$, and hence

$$\nabla\left(\frac{1}{2}\vec{r}\cdot\overset{\rightarrow\rightarrow}{A}\cdot\vec{r}\right) = \vec{r}\cdot\overset{\rightarrow\rightarrow}{A}$$

Clearly

$$\vec{F}_{tide} = -\nabla U_{tide}, \quad U_{tide} = -\frac{1}{2}\vec{r}\cdot\overset{\rightarrow\rightarrow}{A}\cdot\vec{r}$$
$$= \frac{1}{2}GM_{moon}m\;\vec{r}\cdot\frac{1}{d_0^3}(I - 3\hat{d}_0\hat{d}_0)\vec{r} + \text{Const.}$$
$$= \frac{1}{2}\frac{GM_{moon}m}{d_0^2}\frac{r^2}{d_0}\left(1-(3\,\hat{d}_0\cdot\hat{r})^2\right) + \text{Const.}$$

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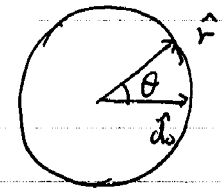


$$\Rightarrow$$

$$U_{tide} = \text{Const.} + \frac{1}{2}\frac{GM_{moon}m}{d_0}\left(\frac{r}{d_0}\right)^2\left(1-3(\hat{d_0}\cdot\hat{r})^2\right)$$

$$= \text{Const.} + \frac{1}{2}\frac{GM_{moon}m}{d_0}\left(\frac{r}{d_0}\right)^2\left(1-3\cos^2\theta\right)$$

$U_{tide} + mgh(\theta) = \text{Const.}$

in which $r = R_e + h(\theta)$.

So

$$\frac{1}{2}\frac{GM_{moon}m}{d_0}\left(\frac{R_e+h(\theta)}{d_0}\right)^2\left(1-3\cos^2\theta\right) + mgh(\theta) = \text{const.}$$

Neglecting $h(\theta)$ in $R_e+h(\theta)$, we have

$$\frac{1}{2}\frac{GM_{moon}m}{d_0}\left(\frac{R_e}{d_0}\right)^2\left(1-3\cos^2\theta\right) + mgh(\theta) = \text{const.}$$

Then

$$\text{Const.} = LHS\left(\theta=\frac{\pi}{2}\right) = \frac{1}{2}\frac{GM_{moon}m}{d_0}\left(\frac{R_e}{d_0}\right)^2 + mg\,h\left(\theta=\frac{\pi}{2}\right).$$

Let $\Delta h(\theta) = h(\theta) - h(\theta=\frac{\pi}{2})$, we have

$$-\frac{3}{2}\frac{GM_{moon}m}{d_0}\left(\frac{R_e}{d_0}\right)^2\cos^2\theta + mg\,\Delta h(\theta) = 0$$

$$\Rightarrow \quad \Delta h(\theta) = +\frac{3}{2}\frac{GM_{moon}R_e^2}{g\,d_0^3}\cos^2\theta$$

$$= +\frac{3}{2}\frac{M_{moon}}{M_{earth}}\left(\frac{R_e}{d_0}\right)^3 R_e\cos^2\theta$$

So

$$h_0 = \frac{3}{2}\,M_{moon}R_e^4 \big/ M_{earth}d_0^3$$

Problem 9.11

(a) $L = T - U$

$$= \frac{1}{2} m \vec{v}_0^{\,2} - U(\vec{r})$$

$$= \frac{1}{2} m (\dot{\vec{r}} + \vec{\Omega} \times \vec{r})^2 - U(\vec{r})$$

(b) Euler-Lagrange eqn. is

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{r}}} - \frac{\partial L}{\partial \vec{r}} = 0$$

$\frac{\partial L}{\partial \dot{\vec{r}}}$ is easy to get:

$$\frac{\partial L}{\partial \dot{\vec{r}}} = \frac{1}{2} m \frac{\partial}{\partial \dot{\vec{r}}} (\dot{\vec{r}} + \vec{\Omega} \times \vec{r})^2$$

$$= \frac{1}{2} m \cdot 2 (\dot{\vec{r}} + \vec{\Omega} \times \vec{r}) = m (\dot{\vec{r}} + \vec{\Omega} \times \vec{r})$$

Hence

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\vec{r}}} \right) = m (\ddot{\vec{r}} + \vec{\Omega} \times \dot{\vec{r}}) .$$

For $\frac{\partial L}{\partial \vec{r}}$, we have

$$\frac{\partial L}{\partial \vec{r}} = \frac{\partial}{\partial \vec{r}} \left(\frac{1}{2} m (\dot{\vec{r}} + \vec{\Omega} \times \vec{r})^2 - U(\vec{r}) \right)$$

$$= \frac{1}{2} m \frac{\partial}{\partial \vec{r}} (\dot{\vec{r}}^2 + 2 \dot{\vec{r}} \cdot (\vec{\Omega} \times \vec{r}) + (\vec{\Omega} \times \vec{r})^2) - \nabla U(\vec{r})$$

By using

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$$

and

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$$

we have

$$2 \dot{\vec{r}} \cdot (\vec{\Omega} \times \vec{r}) = 2 \vec{r} \cdot (\dot{\vec{r}} \times \vec{\Omega})$$

and

$$(\vec{\Omega} \times \vec{r})^2 = (\vec{\Omega} \times \vec{r}) \cdot (\vec{\Omega} \times \vec{r}) = \vec{\Omega} \cdot (\vec{r} \times (\vec{\Omega} \times \vec{r}))$$

$$= \vec{\Omega} \cdot (\vec{\Omega} r^2 - (\vec{\Omega} \cdot \vec{r}) \vec{r}) = \Omega^2 r^2 - (\vec{\Omega} \cdot \vec{r})^2 .$$

So

$$\frac{\partial}{\partial \vec{r}} (\dot{\vec{r}}^2 + 2 \dot{\vec{r}} \cdot (\vec{\Omega} \times \vec{r}) + (\vec{\Omega} \times \vec{r})^2)$$

$$= \frac{\partial}{\partial \vec{r}} (\dot{\vec{r}}^2 + 2 \vec{r} \cdot (\dot{\vec{r}} \times \vec{\Omega}) + \Omega^2 r^2 - (\vec{\Omega} \cdot \vec{r})^2)$$

$$= 2\dot{\vec{r}} \times \vec{\Omega} + 2\Omega^2 \vec{r} - 2(\vec{\Omega}\cdot\vec{r})\,\Omega$$

$$= 2\dot{\vec{r}} \times \vec{\Omega} + 2(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$$

hence

$$\frac{\partial L}{\partial \vec{r}} = m\dot{\vec{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega} \ .$$

Then by Euler-Lagrange eqn.,

$$m(\ddot{\vec{r}} + \vec{\Omega} \times \dot{\vec{r}}) = m\dot{\vec{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega} - \nabla U(\vec{r})$$

i.e.

$$m\ddot{\vec{r}} = -\nabla U(\vec{r}) + 2m\dot{\vec{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$$

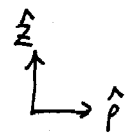
Problem 9.14.

Gravitational potential

$$U_g = -m\vec{g}\cdot\vec{r}$$

Centrifugal potential

$$U_{centr} = -\frac{1}{2}m(\vec{\Omega}\times\vec{r})^2$$



Let's set up a cylindrical coordinate system,

so that

$$\vec{g} = -g\,\hat{z}$$

$$\vec{r} = z\,\hat{z} + \rho\,\hat{\rho} \qquad (\rho = \sqrt{x^2+y^2})$$

$$\vec{\Omega} = \Omega\,\hat{z}$$

Condition for surface of the rotating water to be in equilibrium is that the surface is an equi-potential surface for $U_{tot} = U_g + U_{centri}$.

Let

$$-m\vec{g}\cdot\vec{r} - \frac{1}{2}m(\vec{\Omega}\times\vec{r})^2 = \text{const.}$$

Then

$$mgz - \frac{1}{2}m\left(\Omega\hat{z}\times(\rho\hat{\rho} + z\hat{z})\right)^2 = \text{const.}$$

$$\Rightarrow mgz - \frac{1}{2}m\,\Omega^2\rho^2(\hat{z}\times\hat{\rho})^2 = \text{const.}$$

$$\Rightarrow mgz - \frac{1}{2}m\,\Omega^2\rho^2 = \text{const.}$$

$$\Rightarrow z - \frac{1}{2}\frac{\Omega^2}{g}\rho^2 = \text{const.}'$$

Absorbing const.' into a redefinition of z, we have

$$z = \frac{1}{2}\frac{\Omega^2}{g}\rho^2$$