PHYSICS 110B

HW # 3

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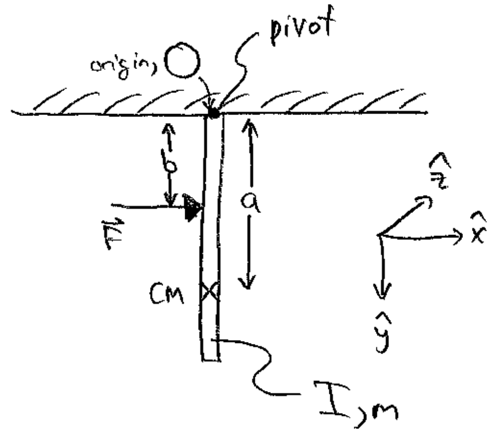

Rod with known mass m, moment of inertia I, and location of center of mass $\vec{r}_{CM}$ is struck sharply with horizontal force F which delivers impulse $F\Delta t = \vec{\xi}$ a distance b below the pivot.

a) Find the rod's angular momentum, $\vec{L}$ and hence momentum, $\vec{P}_{CM}$ just after the impulse:

Initially we have: $\omega_i = P_{CM,i} = 0$

By Euler's equation, and $\vec{\omega} = -\omega\langle 0,0,1\rangle$ due to rotations only in xy plane:

$$\dot{\vec{L}} = \underbrace{\vec{\Gamma}}_{\text{torque due to force } F} \Rightarrow \vec{L} = \vec{\Gamma}\Delta t = \underbrace{\left(b\langle 0,1,0\rangle \times F\langle 1,0,0\rangle\right)}_{-bF\hat{z}}\Delta t = \overleftrightarrow{I}\vec{\omega} = -I\omega\hat{z}$$

where $I \equiv I_{zz}$ and remaining moment of inertia elements don't contribute due to $z=0$ plane of reflection symmetry and rotational constraints $\omega_x = \omega_y = 0$.

Since $F = \vec{\xi}/\Delta t$ we have,

$\Rightarrow \quad \omega = \dfrac{b\xi}{I}$; Hence, $\boxed{\vec{L} = -b\xi\,\hat{z}}$ and since rod is rigid body,

$$\vec{v}_{CM,f} = -\omega\langle 0,0,1\rangle \times a\langle 0,1,0\rangle = \omega a\,\hat{x}$$

$$\therefore \quad \vec{P}_{CM,f} = m\vec{v}_{CM} = \boxed{\frac{mab\xi}{I}\,\hat{x}}$$

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b) Find the impulse η delivered to the pivot:

The total impulse delivered to the rod is due to the striking force F and the reaction force from the pivot,

$$\Delta \vec{p} = \xi \hat{x} + \vec{\eta} = \vec{P}_{cm,f} - \cancel{\vec{P}_{cm,i}}^{\,0} = \frac{mab\xi}{I} \hat{x}$$

$$\Rightarrow \boxed{\vec{\eta} = \left(\frac{mab}{I} - 1\right) \xi \hat{x}}$$

c) Find value of b (ie. b_o) for which $\eta = 0$:

$\Rightarrow$ by inspection, $\boxed{b_o = \frac{I}{ma}}$ which defines the "sweet spot" of the rod

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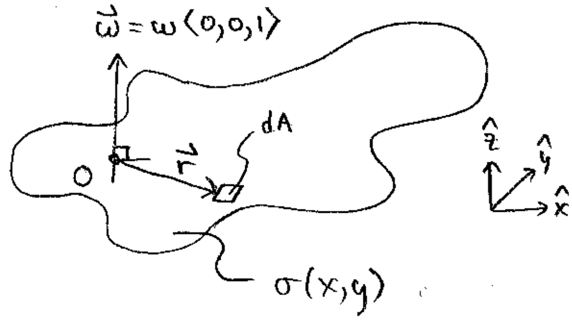
Consider a rigid plane body (i.e. lamina) such as a flat piece of sheet metal rotating about a point O in the body. Choosing axes so that lamina lies in xy plane,



(i) Find vanishing elements of the inertia tensor $\overleftrightarrow{I}$ at pt. O:

Since $z=0$ for all differential masses of area dA, all products of inertia (i.e. off-diagonal elements of $\overleftrightarrow{I}$) involving a factor of z automatically vanish:

$$\Rightarrow \quad I_{xz} = -\int \underbrace{dm}_{=\sigma(x,y)\,dx\,dy}\, x \overset{0}{\overset{\shortparallel}{z}} = 0 = I_{zx}\text{, by symmetry of } \overleftrightarrow{I}$$

$$\Rightarrow \quad I_{yz} = -\int dm\, yz = 0 = I_{zy}$$

$$\therefore \quad \boxed{I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0}$$

Note: This also follows by recognizing that $z=0$ is a reflection plane of symmetry of the lamina at pt. O.

(ii) Prove that $I_{zz} = I_{xx} + I_{yy}$ by writing both sides:

$$\Rightarrow \quad I_{zz} = \int dx\,dy\,\sigma(x,y)(x^2+y^2) \overset{?}{=} \int dx\,dy\,\sigma(x,y)(y^2+\cancel{z^2}^{\,0}) + \int dx\,dy\,\sigma(x,y)(\cancel{z^2}^{\,0}+x^2)$$

due to $z=0$ for all differential masses

$$\therefore \text{ by inspection we have } \boxed{I_{zz} = I_{xx} + I_{yy}}$$

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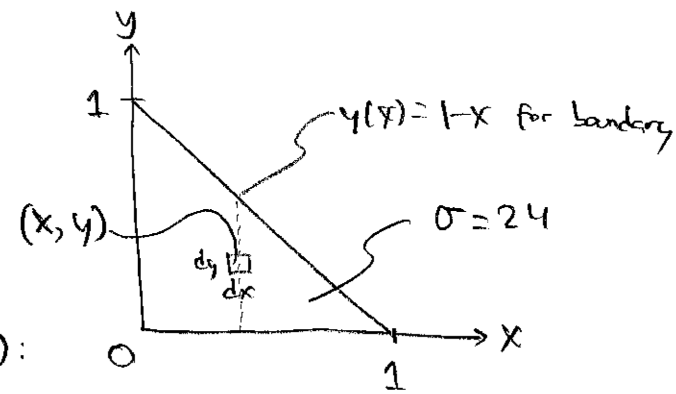
A thin, flat, uniform metal triangle lies in the xy plane with its corners as shown. Its surface density (mass/area) is $\sigma = 24$.



a) Find triangle's inertia tensor, $\overleftrightarrow{I}$ (about pt. 0):

$$\Rightarrow I_{xx} = \sigma \int_0^1 dx \underbrace{\int_0^{1-x} dy (y^2 + \cancel{z^2}^{\,0})}_{\frac{y^3}{3}\Big|_0^{1-x} = \frac{1}{3}(1-x)^3} = -\frac{\sigma}{3}\frac{(1-x)^4}{4}\Big|_0^1 = \frac{\sigma}{12} = I_{yy}, \text{ by symmetry}$$

$$\Rightarrow I_{zz} = \sigma \int_0^1 dx \int_0^{1-x} dy\,(x^2+y^2) = I_{xx} + I_{yy} = 2 I_{xx} = \frac{\sigma}{6}$$

Off-diagonal elements:

- We recognize that $z=0$ is plane of reflection symmetry; thus all products of inertia associated with z-axis vanish:

$$\Rightarrow I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0$$

$$\Rightarrow I_{xy} = -\sigma \int_0^1 dx \underbrace{\int_0^{1-x} dy\, xy}_{x\frac{y^2}{2}\Big|_0^{1-x} = \frac{1}{2}x(1-x)^2} = -\frac{\sigma}{2}\underbrace{\int_0^1 dx\; x^3 - 2x^2 + x}_{\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\Big|_0^1 = \frac{3}{4} - \frac{2}{3} = \frac{9-8}{12} = \frac{1}{12}} = -\frac{\sigma}{24}$$

$\therefore$ Substituting $\sigma = 24$ we obtain:

$$\boxed{\overleftrightarrow{I} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}}$$

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b) Find principal moments (i.e. eigenvalues) and corresponding axes (i.e. eigenvectors):

$\Rightarrow$ Solve the secular eqn: $\overleftrightarrow{I}\vec{\omega} = \lambda\vec{\omega} \Rightarrow (\overleftrightarrow{I} - \lambda \overleftrightarrow{\mathbb{1}})\vec{\omega} = 0$ ← identity matrix

$\Rightarrow \det(\overleftrightarrow{I} - \lambda\overleftrightarrow{\mathbb{1}}) = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0 \; ; \Rightarrow (2-\lambda)(2-\lambda)(4-\lambda) + (-1)(-1)(4-\lambda) = 0$$

$$\Rightarrow (4-\lambda)\underbrace{[\lambda^2 - 4\lambda + 4 - 1]}_{(\lambda-3)(\lambda-1)} = 0$$

$\therefore$ principal moments: $\boxed{\begin{matrix}\lambda_1 = 1 \\ \lambda_2 = 3 \\ \lambda_3 = 4\end{matrix}}$

Find corresponding axes by solving equations for each λ.
Here we use row reduction, whereby we can perform the following 3 row operations on a matrix:

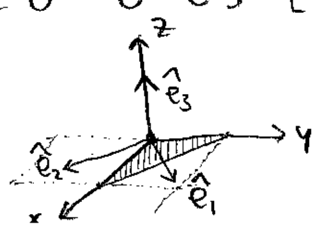
- multiply row by a constant (nonzero)
- Switching two rows
- adding a constant multiple of one row to another row

$$\Rightarrow \text{for } \lambda_1: \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} \omega_1 = \omega_2 \\ \omega_3 = 0 \end{cases} \quad \therefore \boxed{\hat{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}}$$

where we have normalized the eigenvector

$$\Rightarrow \text{For } \lambda_2: \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} \omega_1 = -\omega_2 \\ \omega_3 = 0 \end{cases} \quad \therefore \boxed{\hat{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}}$$

$$\Rightarrow \text{For } \lambda_3: \begin{bmatrix} -2 & -1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} 2\omega_1 = \omega_2 = 0 \\ \omega_2 = 0 \end{cases} \quad \therefore \boxed{\hat{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}}$$



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Suppose that the moment of inertia tensor $\overleftrightarrow{I}$ of a rigid body is known, and consequently so are its 3 principal axes (directions $\hat{e}_1, \hat{e}_2, \hat{e}_3$) and corresponding principal moments $\lambda_1, \lambda_2, \lambda_3$.

a) Prove that if $\lambda_i \neq \lambda_j$, it follows that $\hat{e}_i \cdot \hat{e}_j = 0$.

Using matrix multiplication notation to represent a dot product of two vectors:

$$\vec{a} \cdot \vec{b} \leftrightarrow \underbrace{\tilde{\underline{a}}}_{\text{the matrix transpose of column vector } \underline{a}} \underline{b} = (a_x, a_y, a_z) \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$$

we evaluate the scalar quantity, $\tilde{\underline{e}}_i \, \underline{I} \, \underline{e}_j$ in 2 ways, given that $\hat{e}_i, \hat{e}_j$ are eigenvectors of $\overleftrightarrow{I}$:

- right multiplication:

$$\tilde{\underline{e}}_i \left(\underbrace{\underline{I} \, \underline{e}_j}_{= \lambda_j \underline{e}_j} \right)$$

- left multiplication:

$$\left(\tilde{\underline{e}}_i \, \underline{I} \right) \underline{e}_j$$

$$\tilde{\underline{e}}_i \, \underline{I} = \widetilde{\left(\tilde{\underline{I}} \, \underline{e}_i \right)} = \widetilde{\left(\lambda_i \underline{e}_i \right)} = \lambda_i \tilde{\underline{e}}_i$$

$\tilde{\underline{I}} = \underline{I}$, since symmetric tensor

where we have used identity $\widetilde{(\underline{A}\,\underline{B})} = \tilde{\underline{B}} \tilde{\underline{A}}$

Combining both evaluations, and using $\lambda_i \neq \lambda_j$:

$$\Rightarrow \lambda_j \underbrace{\tilde{\underline{e}}_i \underline{e}_j}_{\equiv \hat{e}_i \cdot \hat{e}_j} = \lambda_i \tilde{\underline{e}}_i \underline{e}_j \text{, which holds iff } \boxed{\hat{e}_i \cdot \hat{e}_j = 0}$$

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b) Prove that if $\lambda_1 \neq \lambda_2 \neq \lambda_3$, then the directions of the 3 principal axes are uniquely determined:

$\Rightarrow$ By results of previous part, it follows that all three eigenvectors are mutually orthogonal.

Now suppose that $\hat{e}_1'$ was a different eigenvector from the original $\hat{e}_1$, but with same eigenvalue λ_1. By part (a), both $\hat{e}_1'$ and $\hat{e}_1$ have to be orthogonal to $\hat{e}_2$ and $\hat{e}_3$. Thus, $\hat{e}_1$ and $\hat{e}_1'$ must point along the same direction and so the directions of the 3 principal axes $\boxed{\text{are uniquely determined.}}$

c) Prove that if only two of the principal moments are equal, $\lambda_1 = \lambda_2$ say, then the corresponding principal axes are not uniquely determined:

$\Rightarrow$ If $\lambda_1 = \lambda_2 = \lambda$, the proof of orthogonality in part (a) breaks down so that generally, $\hat{e}_1 \cdot \hat{e}_2 \neq 0$.

If $\vec{a}$ is any vector in the plane of $\hat{e}_1$ and $\hat{e}_2$, then we can write:

$$\vec{a} = \alpha \hat{e}_1 + \beta \hat{e}_2 \text{ , where } \alpha, \beta \text{ are constants}$$

$$\Rightarrow \underline{I}\,\underline{a} = \underline{I}(\alpha \underline{e}_1 + \beta \underline{e}_2) = \alpha \underbrace{\underline{I}\,\underline{e}_1}_{\lambda \underline{e}_1} + \beta \underbrace{\underline{I}\,\underline{e}_2}_{\lambda \underline{e}_2} = \lambda \underline{a}$$

Thus we see that any vector in the plane of $\hat{e}_1$ and $\hat{e}_2$ is an eigenvector of $\underline{I}$ with the same eigenvalue λ, so that any direction in the plane is a principal axis with same principal moment. Hence, the corresponding principal axes $\boxed{\text{are not uniquely determined.}}$

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d) Prove that if all three principal moments are equal, then any axis is a principal axis with the same principal moment, λ:

$\Rightarrow$ Any vector $\vec{a}$ can be written as a linear combination of the three independent vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$. If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, then the same argument just given shows that $\vec{a}$ is an eigenvector with the same eigenvalue. That is, $\boxed{\text{any direction}}$ is a principal axis with the same principal moment.