Fermi liquid theory (II) Renormalization to physical quantities, and Pomeranchuk instability

§ Fermi liquid corrections to physical ~~others~~ quantities.

dimensionless Landau interaction function

$$f_{s,a}(\cos\theta) = \sum_l f_{l;\,s,a} \, P_l(\cos\theta)$$

$$F_{s,a} = N_0 \, f_{l;s,a} \; ; \; N_0 \text{ density of state}$$

The interaction effects are summarized in the two sets of Landau parameters.

☆ S-wave channel : molecular method

Spin-susceptibilies :

$$f_0^a \, \sigma\sigma' = N^{-1}(0) \, F_0^a \, \sigma\sigma'$$

$$\delta \mathcal{E}^{(2)} = \frac{1}{2} N^{-1}(0) \, F_0^a \sum_{pp'\sigma\sigma'} \sigma\sigma' \, \delta n_{p\sigma} \, \delta n_{p\sigma'} = \frac{1}{2} N^{-1}(0) \, F_0^a \, (S_z)^2$$

define molecule field $\quad E = -\int \vec{h}_{mol} \cdot d\vec{S}$

$$\Rightarrow h_{mol}(S) = -\frac{\delta \mathcal{E}}{\delta S_z} = -N^{-1}(0) \, S_z \, F_0^a$$

$$h_{tot} = h_{ex} + h_{mol} = h_{ex} - N^{-1}(0) \, S_z \, F_0^a$$

$$S_z = \chi_0 \, h_{tot} = \chi_0 \, h_{ext} - \chi_0 N^{-1}(0) \, S_z \, F_0^a$$

$$S_z (1 + \chi_0 F_0^a N^{-1}(0)) = \chi_0 \, h_{ext} \quad \Rightarrow \quad \boxed{\chi = \frac{\chi_0}{1 + \chi_0 F_0^a (N(0))^{-1}}}$$

Compressibility $\quad f_0^s \cancel{}' = N^{-1}(0)\, F_0^s$

$$\delta \mathcal{E}^{(2)} = \frac{N^{-1}(0)}{2} F_0^s \sum_p \delta n_p \delta n_{p'} = \frac{1}{2} (N(0))^{-1} F_0^s (\delta n)^2$$

$$h_{mol} = - N^{-1}(0)\, F_0^s\, \delta n \quad \Rightarrow \quad \boxed{\frac{dn}{d\mu} = \frac{N(0)}{1+F_0^s}}$$

☆ p-wave channel : effective mass.

define $\quad n(r,t) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3}\, n_{p,\sigma}(r,t)$

$\quad \vec{j}(r,t) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3}\, \vec{\nabla}_p \mathcal{E}_{p\sigma}(r,t)\, n_{p\sigma}(r,t)$

allow a slow spatial variation.

linearizing the expression of $\vec{j}(r,t)$, by using

$$\mathcal{E}_{p\sigma}(r,t) = \mathcal{E}_p^0 + \int \frac{d^3p'}{(2\pi)^3} f^s_{\sigma\sigma'}(p p')\, \delta n_{p'\sigma'}(r,t)$$

$$n_{p\sigma}(r,t) = n_p^0 + \delta n_{p,\sigma}(r,t)$$

$$\vec{j}(r,t) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3} \nabla_p \mathcal{E}^0_{p\sigma}\, \delta n_{p\sigma}(r\,t) + \nabla_p \delta \mathcal{E}_{p\sigma}(r\,t) . n_p^0$$

↙ partial derivative

$$= \sum_\sigma \int \frac{d^3p}{(2\pi)^3} \nabla_p \mathcal{E}_p^0\, \delta n_{p\sigma}(r,t) - \nabla_p n_p^0\, \delta \mathcal{E}_{p\sigma}(r,t)$$

$$= \sum_\sigma \int \frac{d^3p}{(2\pi)^3} v_p \left[\delta n_{p\sigma}(r,t) - \frac{\partial n^0_{p\sigma}}{\partial \mathcal{E}_p} \int \frac{d^3p'}{(2\pi)^3} f_{\sigma\sigma'}(p p')\, \delta n_{p'\sigma}(r t) \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} v_p\, \delta n_p(r,t) + \int \frac{d^3p}{(2\pi)^3} v_p \left(-\frac{\partial n^0_{p\sigma}}{\partial \mathcal{E}_p}\right) \int \frac{d^3p'}{(2\pi)^3} f^s(p p')\, \delta n_{p'}(r t)$$

$$\int \frac{d^3p}{(2\pi)^3}\, \vec{v}_p \left(-\frac{\partial n^0_{p\sigma}}{\partial \varepsilon_p}\right) f^s(p,p') = N(0)\int \frac{d\Omega}{4\pi} \sum_l f^s_l\, P_l(\cos\theta)\, v_F \cos\theta\ \hat{z}$$

↳ set p' along z-axis in the p-space

$$= \frac{N(0)}{3} f_1^s\, v_F\, \hat{z},$$

{ other two direction average to zero

$$\Rightarrow \int \frac{d^3p}{(2\pi)^3}\, \vec{v}_p \left(-\frac{\partial n^0_{p\sigma}}{\partial \varepsilon_p}\right) f^s(\vec{p},\vec{p}\,') = \frac{N(0)}{3} f_1^s\, \vec{v}_{p'} = \frac{F_1^s}{3}\, \vec{v}_{p'}$$

$$\vec{j}(r\,t) = \int \frac{d^3p}{(2\pi)^3}\, \vec{v}_p\, \delta n_p(r\,t) + \frac{F_1^s}{3}\int \frac{d^3p'}{(2\pi)^3}\, \vec{v}_{p'}\, \delta n_{p'}(r\,t)$$

$$= \int \frac{d^3p}{(2\pi)^3}\, \vec{v}_p \left(1+\frac{F_1^s}{3}\right) \delta n_p(r,t) = \int \frac{d^3p}{(2\pi)^3}\, \frac{\vec{P}}{m^*}\left(1+\frac{F_1^s}{3}\right) \delta n_p(r,t)$$

on other hand, by adiabatic continuity

$$\vec{j}(r\,t) = \int \frac{d^3p}{(2\pi)^3}\, \frac{\vec{P}}{m}\, \delta n_p(r,t) \Rightarrow \boxed{\frac{1}{m} = \frac{1}{m^*}\left(1+\frac{F_1^s}{3}\right)}$$

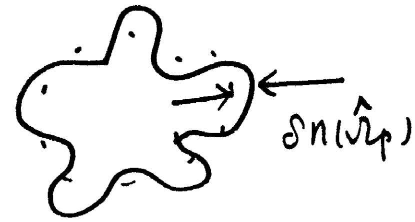
Similarly, we can derive spin current

$$j_i^\mu = 2\int \frac{d^3p}{(2\pi)^3}\left(1+\frac{F_1^a}{3}\right)\frac{P_i}{m^*}\ \sigma_p^\mu(r,t)$$

we can define spin-effective mass $\dfrac{1}{m_s^*} = \dfrac{1}{m^*}\left(1+\dfrac{F_1^a}{3}\right)$

$$\boxed{\frac{m_s^*}{m} = \frac{1+\frac{1}{3}F_1^s}{1+\frac{1}{3}F_1^a}}$$

§ For general channels $F^{a,s}_{\ell}$



$$\delta n = V \int \frac{p^2 dp}{(2\pi)^3} \int d\Omega_p \, \delta n(p, \Omega_p) = V \int d\Omega \, \delta n(\hat{\Omega}_p)$$

where $\delta n(\hat{\Omega})$ is defined as $\int \frac{p^2 dp}{(2\pi)^3} \delta n(p, \Omega_p)$, i.e. integrate over radius direction.

we expand the angular distribution in terms of harmonic oscillators

$$\delta n(\hat{\Omega}_p) = \sum_{\ell m} \delta n_{\ell m} Y_{\ell m}(\hat{\Omega}_p)$$

$$E^{(2)} = \frac{1}{2V} \sum_{pp'} f_{\sigma\sigma'}(\hat{p}\,\hat{p}') \, \delta n_{p\sigma} \, \delta n_{p'\sigma'} = \frac{V}{2} \int d\Omega_p \, d\Omega_{p'} \, f_{\sigma\sigma'}(p\,p') \, \delta n_\sigma(\Omega_p) \, \delta n_{\sigma'}(\Omega_p')$$

$$= \frac{V}{2} N^{-1}(0) \int d\Omega_p \, d\Omega_{p'} \underline{\sum_{\ell m} F^s_\ell \frac{4\pi}{2\ell+1} Y^*_{\ell m}(\Omega_p) Y_{\ell m}(\Omega_{p'})} \leftarrow \text{addition theorem}$$

$$\Big[\Big(\sum_{\ell_1 m_1} Y_{\ell_1 m_1}(\Omega_p) \, \delta n^s_{\ell_1 m_1}\Big) \Big(\sum_{\ell_2 m_2} Y_{\ell_2 m_2}(\Omega_p') \, \delta n^s_{\ell_2 m_2}\Big)$$

$$+ (s \to a) \Big]$$

where $F_{\sigma\sigma'} = F^s + F^a \sigma\sigma'$, $\delta n_{s,a} = \delta n_\uparrow \pm \delta n_\downarrow$

$$E^{(2)} = \frac{V}{2} N^{-1}(0) \Big[\sum_{\ell m} F^s_\ell \frac{4\pi}{2\ell+1} \delta n^{*(s)}_{\ell m} \, \delta n^{(s)}_{\ell m} + (s \to a) \Big]$$

The kinetic energy increase

$$\delta E^{(1)} = \sum \varepsilon_p \delta n_p = V \int d\Omega \int \frac{p^2 dp}{(2\pi)^3} \varepsilon_p \, \delta n(p, \Omega_p)$$

$$\int \frac{p^2 dp}{(2\pi)^3} \varepsilon_p \delta n(p,\hat{\Omega}_p) = \frac{p_F^2}{(2\pi)^3} v_F \cdot \frac{1}{2} (\delta p_F)^2 \leftarrow \quad \begin{array}{l} \varepsilon_p = v_F \cdot \delta p \\ p^2 \to p_F^2 \end{array}$$

Compare with $\int \frac{p^2 dp}{(2\pi)^3} \delta n(p,\hat{\Omega}_p) = \frac{p_F^2}{(2\pi)^3} \delta p_F = \delta n(\Omega_p)$

$$\Rightarrow \int \frac{p^2 dp}{(2\pi)^3} \varepsilon_p \delta n(p,\hat{\Omega}_p) = \frac{v_F}{2} [\delta n(\Omega_p)]^2 \Big/ \frac{p_F^2}{(2\pi)^3} = 4\pi N^{-1}(0) [\delta n(\Omega_p)]^2$$

$$\delta E^{(1)} = V N^{-1}(0) \int d\Omega [\delta n(\Omega_p)]^2 = 2\pi V N^{-1}(0) \sum_{lm} |\delta n^s_{lm}|^2 + |\delta n^a_{lm}|^2$$

$$\Rightarrow \quad \Delta E = 2V N^{-1}(0) \sum_{lm} \left\{ \left(1 + \frac{F^s_l}{2l+1}\right) |\delta n^s_{lm}|^2 + (s \to a) \right\}$$

From thermodynamic properties, we know

$$\Delta E = \sum_{lm} \frac{1}{2\chi^s_l} |\delta n^s_{lm}|^2 + (s \to a)$$

$$\Rightarrow \quad \frac{1}{\chi_l{}^{s,a}_{FL}} = \frac{1}{\chi^{s,a}_{l,0}} \left(1 + \frac{F^{s,a}_l}{2l+1}\right)$$

i.e. $$\boxed{\chi^{s,a}_{FL,l} = \frac{\chi^{s,a}_{l,}}{1 + \frac{F^{s,a}}{2l+1}}}$$

in ^3He $\quad F^s_0 \simeq 10.8. \quad F^0_a \approx -0.75$

Compressibility is greatly reduced

spin-susceptibility is greatly enhanced!

③

§: effective mass renormalization

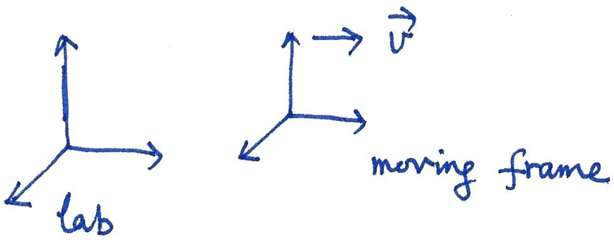
a moving frame with velocity $\vec{V}$

Consider that we do a Galilean transformation,

$$H = \sum_i \frac{P_i^2}{2m} \rightarrow \sum_i \frac{(\vec{P}_i + m\vec{V})^2}{2m}$$

where P_i is momentum in the moving frame

$P_i + m\vec{V}$ is the momentum in the lab frame.



P_i is cannonical momentum

$P_i + mV$ is mechanical momentum

In the lab frame, the current reads

$j(r,t) = \sum \delta(r - r_i)\,(\vec{P}_i + m\vec{V})$, which is zero because the system remains at rest.

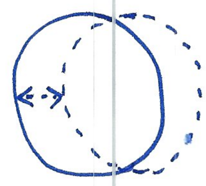
$$\langle j \rangle = \frac{1}{m}\langle p(t)\rangle + \vec{V} = 0 \Rightarrow \langle \vec{P}_i \rangle = -\vec{V}m$$

total mometum $\langle \vec{Q} \rangle = -Nm\vec{V}$ in the moving frame.

Now let us calculate $\langle \vec{Q} \rangle$ in the moving frame by another method.

Let us conside $-\vec{v}$ as an external field, and $\vec{Q}$ as the response

$$\langle \vec{Q} \rangle = -\frac{\delta E}{\delta \vec{v}}$$



$\delta p = -m\vec{v}$

④

$$\langle \vec{Q} \rangle = \sum_{\sigma} p\, \delta n_{p\sigma} = V \cdot P_F \int d\Omega\, \Omega_p \int \frac{dp}{(2\pi)^3} P_F^2\, \delta n_{p\sigma} = V \cdot P_F \int d\Omega_p \cos\theta_p\, \delta n(\Omega_p)$$

$$= V \cdot P_F \sqrt{\frac{4\pi}{3}}\, \delta n_{10}^{s}$$

$$E(\vec{v}) = E(\vec{v}=0) + V P_F \left(\sqrt{\frac{4\pi}{3}}\, v\right) \delta n_{10}^{s}$$

$$\frac{E(\vec{v})}{V} = 2\pi N^{-1}(0)\left(1+\frac{F_1^S}{3}\right)(\delta n_{10}^{s})^2 + P_F\left(\sqrt{\frac{4\pi}{3}}\, v\right)\delta n_{10}^{s}$$

$$\Rightarrow \langle \delta n_{10}^{s} \rangle = -\frac{N(0)\, P_F \left(\sqrt{\frac{4\pi}{3}}\right)}{4\pi\left(1+\frac{F_1^S}{3}\right)}\, v = -\frac{k_F^2\, P_F}{4\pi^3 \hbar\, v_F}\, \frac{\sqrt{\frac{4\pi}{3}}}{1+\frac{F_1^S}{3}}\, v$$

$$\langle Q \rangle = -V \frac{P_F}{\hbar}\, \frac{k_F^2\, m^* v}{3\pi^2\left(1+\frac{F_1^S}{3}\right)} = -V \frac{k_F^3}{3\pi^2}\, \frac{m^* v}{1+\frac{F_1^S}{3}} = -\frac{N m^* \boldsymbol{v}}{1+\frac{F_1^S}{3}}$$

$$\Rightarrow \quad m = \frac{m^*}{1+\frac{F_1^S}{3}} \qquad \text{i.e.} \quad \boxed{\frac{m^*}{m} = 1 + \frac{F_1^S}{3}}$$

§ Pomeranchuk instability

Consider the Fermi surface as an elastic membrane in momentum space. The deformation of the Fermi surface not only changes the Kinetic energy, but also changes the interaction energy.

As we showed before,

$$\delta E \propto \left(1+\frac{F_\ell^{s,a}}{2\ell+1}\right)|\delta n_{\ell m}^{s,a}|^2 + O(\delta n_{\ell m}^{s})^4 + \cdots$$

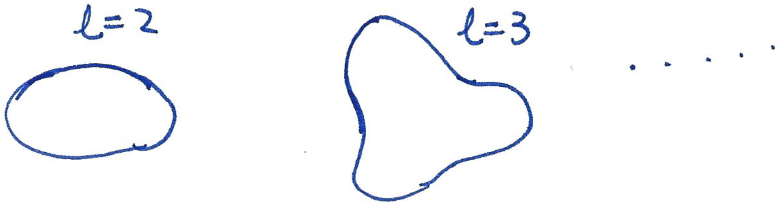
if $F_\ell^{s.a} < -(2\ell+1)$, then the Fermi surface will not be spheric stable, but develop distortions.

$F_0^s \rightarrow$ phase seperation : diveragence of Compressibility

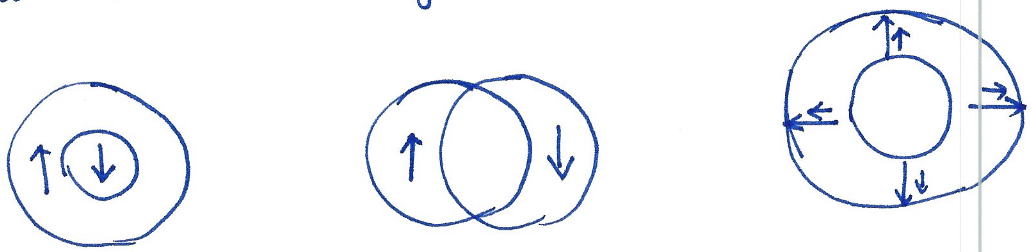
$F_0^a \rightarrow$ Ferromagnetism : divergence of spin-susceptibility

for $\ell \geq 1$, Fermi surface anisotropic distortions.

electronic liquid crystal phase



* Unconventional magnetism



F_0^a F_1^a

p-wave magnetism

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