Mid term Phys 211 B

1. In class, we have already learned the Lindhard response function

$$\chi_0(q,\omega) = \frac{2}{V}\sum_k \frac{n_{k+q} - n_k}{\hbar\omega - (\varepsilon_{k+q} - \varepsilon_k) + i\eta},$$

and have evaluated it in 3D electron systems at the long wave length limit. Now we consider the situation at 2D.

a) at $T = 0K$, and in the limit of $v_F q \ll \varepsilon_F$, we can express

$$\chi_0^{2D}(q,\omega) = N(0)\, f(s),$$

where $s = \omega / v_F q$ is a dimensionless parameter, and $N(0)$ is the density of states at the Fermi surface.

Show that

$$f(s) = \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{-\cos\theta}{s - \cos\theta + i\eta},$$

where θ is the azimuthal angle of momentum k on the Fermi surface.

b) Evaluate the imaginary part of $\chi_0^{2D}(q,\omega)$ at $v_F q \ll \varepsilon_F$, as a function of $s = \frac{\omega}{v_F q}$. Plot the result. What's the qualitative differences between $\mathrm{Im}\,\chi_0(q,\omega)$ at 3D and 2D?

c) The 2D Fourier transform of the Coulomb interaction $\frac{e^2}{r}$ is $\frac{2\pi e^2}{q}$. By evaluating the 2D di-electric function (longitudinal)

$$\varepsilon(q,\omega) = 1 + \frac{2\pi e^2}{q} \chi_0^{2D}(q,\omega),$$

and finding its zero point, we can find the dispersion relation of 2D plasmons at the long wave length limit. The density response $\chi^{2D}(q,\omega)$ after taking care of the Coulumb interaction at the RPA level is $\chi^{2D}(q,\omega) = \chi_0^{2D}(q,\omega)/\varepsilon(q,\omega)$.

Prove that the 2D plasmon is gapless, and its dispersion relation is proportional to $\sqrt{q}$. Find the dispersion relation of $\omega_{2D}(q)$.

d) What is the temperature-dependence, as $T \to 0$, of the contribution of the 2D plasmon to the specific heat?

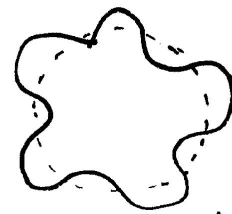
e) Consider a bilayer system with two identical planes spaced by a distance d. Find the frequencies $\omega(q)$ of the two branches of plasmon like modes in the long wave length limit with $qd \ll 1$.

Hint: the Fourier transform of the inter-plane Coulumb potential $\frac{e^2}{\sqrt{r^2+d^2}}$ is $\frac{2\pi e^2}{q} e^{-qd}$.

2. Pomeranchuk instabilities of Fermi liquids

Consider Fermi surface as an elastic membrane in momentum space. Let us distarb it a little bit by creating $\delta n_\sigma(\vec{p})$.

Let us define the angular deformation by integrating out the radius direction



$$\delta n_\sigma(\hat{\Omega}_p) = \int \frac{p^2 dp}{(2\pi)^3} \, \delta n_\sigma(p, \Omega_p),$$ where Ω_p is the solid angle direction of $\hat{p}$.

Then we define the the spherical-harmonic components of $\delta n(\hat{\Omega}_p)$ as

$$\delta n_\sigma(\hat{\Omega}_p) = \sum_{lm} \delta n_{lm\sigma} \, Y_{lm}(\hat{\Omega}_p).$$

a) Prove the cost of the Kinetic energy due to the Fermi surface deformation is

$$\frac{\delta E^{(1)}}{V} = 2\pi N^{-1}(0) \sum_{lm} \left[|\delta n^s_{lm}|^2 + |\delta n^a_{lm}|^2 \right]$$

where $\delta n^{s,a}_{lm} = \delta n_{lm\uparrow} \pm \delta n_{lm\downarrow}$, $N(0)$ is the density of states at Fermi surface

b) Prove that the change of the inter-action energy is

$$\frac{\delta E^{(2)}}{V} = 2\pi N^{-1}(0) \sum_{lm} \frac{F^s_l}{2l+1} |\delta n^s_{lm}|^2 + \frac{F^a_l}{2l+1} |\delta n^a_{lm}|^2$$

c) Add two contributions together, and show that Fermi surface will not be stable if $F^{s,a}_{\ell} \leq -(2\ell+1)$, which is called Pomeranchuck instabitity. Ferromagnetism is one of the simplest version of Pomeranchuck instability, which is in the F^{a}_{0} channel. Draw the Fermi surface shape after Ferromgnetic instability occurs. Can you imagine what will happen on the Fermi surface structure if Pomeranchuk instabity occurs at channels of $(\ell \geq 1)$?

(The F^{s}_{1} channel is subtle, and you can neglect it).

3) Spin waves in ferri-magnets.

Ferri-magnets mean a two-sublattic system with different spin-values S_A and S_B coupled by anti-ferro magnetic exchange $J > 0$. Conside the cubic lattice.

$$H = J \sum_{\substack{i, \\ \delta = \pm\hat{x}, \pm\hat{y}, \pm\hat{z}}} S_i^A \cdot S_{i+\delta}^B . \quad \text{with } S^A > S^B .$$

a) Develop the H-P spin-wave theory for 3D-ferri-magnetic system.

Show that at long wave length limit, there are two branches of spin-wave excitations with dispersions of

$$\omega_k^{\pm} = \begin{cases} 2zJ(S_a - S_b) + 4Ja^2 \dfrac{S_a S_b}{S_a - S_b} k^2 \\ 4Ja^2 \dfrac{S_a S_b}{S_a - S_b} k^2 \end{cases}$$

, ($z = 6$ is the coordination number).

a is the lattice constant.

b) Show that the zero-point fluctations of S_A^z and S_B^z :

$$\langle \Delta S_A^z \rangle = \langle \Delta S_B^z \rangle = \int_{BZ} \frac{1}{2} \left\{ (1 - c^2 \gamma_k^2)^{-1/2} - 1 \right\} \frac{dk}{(2\pi)^2},$$

where $\gamma_k = z^{-1} \sum_{\delta = \pm\hat{x}, \pm\hat{y} \pm\hat{z}} e^{i\vec{k}\cdot\vec{\delta}}$, $C = \dfrac{2(S_a S_b)^{1/2}}{S_a + S_b}$,

$$\Delta S_A^z = S_A - \langle S_A^z \rangle , \text{ and } \Delta S_B^z = S_B + \langle S_B^z \rangle$$