Solution to HW 2

1.1 $G_r(t-t') = \frac{-i}{\hbar}\theta(t-t')\langle|[A(t), B(t')]|\rangle$

$$= -\frac{i}{\hbar}\theta(t-t') Z^{-1}\sum_m \{\langle m|A(t)B(t')|m\rangle - \langle m|B(t')A(t)|m\rangle\} e^{-\beta E_m}$$

$$= -\frac{i}{\hbar}\theta(t-t') Z^{-1}\sum_{m,n}\{\langle m|A(t)|n\rangle\langle n|B(t')|m\rangle - \langle m|B(t')|n\rangle\langle n|A(t)|m\rangle\} e^{-\beta E_m}$$

$$A(t) = e^{iHt/\hbar} A e^{-iHt/\hbar}, \quad B(t') = e^{iHt'/\hbar} B e^{-iHt'/\hbar}$$

$$\Rightarrow G_r(t-t') = -\frac{i}{\hbar}\Theta(t-t') Z^{-1}\sum_{m,n}\Big\{e^{i(E_m-E_n)t/\hbar}\cdot e^{+i(E_n-E_m)t'/\hbar}\langle m|A|n\rangle\langle n|B|m\rangle$$

$$- e^{i(E_m-E_n)t'/\hbar} e^{+i(E_n-E_m)t/\hbar}\langle m|B|n\rangle\langle n|A|m\rangle\Big\} e^{-\beta E_m}$$

$$= -\frac{i}{\hbar}\Theta(t-t') Z^{-1}\sum_{m,n} e^{i(E_m-E_n)(t-t')/\hbar}\{\langle m|A|n\rangle\langle n|B|m\rangle\}\{e^{-\beta E_m} - e^{-\beta E_n}\}$$

$$= \frac{-i}{\hbar}\Theta(t-t') Z^{-1}\sum_{m,n} e^{-\beta E_m} e^{i(E_m-E_n)(t-t')/\hbar}\underbrace{[1 - e^{\beta(E_n-E_m)}]}_{\langle m|A|n\rangle\langle n|B|m\rangle}$$

or exchange m, n

$$\rightarrow -\frac{i}{\hbar}\Theta(t-t') Z^{-1}\sum_{m,n} e^{-\beta E_m}\langle m|B|n\rangle\langle n|A|m\rangle \cdot$$

$$e^{-\frac{i}{\hbar}(E_m-E_n)t}[e^{\beta(E_m-E_n)} - 1]$$

$$G_r(\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t}\cdot G_r(t)$$

$$= \frac{1}{Z} \sum_{m,n} e^{-\beta E_m} \langle m|B|n\rangle\langle n|A|m\rangle \left(e^{\beta(E_m - E_n)} - 1\right)$$

$$\int_{-\infty}^{+\infty} dt \ \left(\frac{-i}{\hbar}\right) \theta(t) \ e^{i\left(\omega - \frac{(E_m - E_n)}{\hbar} + i\eta\right) t}$$

$$= Z^{-1} \sum_{m,n} e^{-\beta E_m} \langle m|B|n\rangle\langle n|A|m\rangle \frac{e^{\beta(E_m - E_n)} - 1}{\omega - (E_m - E_n)/\hbar + i\eta}$$

using the fact $\dfrac{1}{x \pm i\eta} = P\left(\dfrac{1}{x}\right) \mp i\pi\delta(x)$

$$J = -2\,\mathrm{Im}\, G_r(\omega) = Z^{-1} (2\pi\hbar) \sum_{m,n} e^{-\beta E_m} \langle m|B|n\rangle\langle n|A|m\rangle \left(e^{\beta(E_m - E_n)} - 1\right) \delta\left(\omega - (E_m - E_n)\right)$$

$\Rightarrow$ Since $J(\omega)$ is a sumation over δ-function.

It's easy to show

$$G_r(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{J(\omega')}{\omega - \omega' + i\eta} d\omega'$$

1.2 in the case of $A = B$, similarly to the proof of (1.1)

$$S(t-t') = Z^{-1} \sum_{m,n} \langle m| A(t) |n\rangle\langle n| A(t') |m\rangle \, e^{-\beta E_m}$$

$$= Z^{-1} \sum_{m,n} e^{\frac{i}{\hbar}(E_m - E_n)(t-t')} e^{-\beta E_m} \left|\langle m|A|n\rangle\right|^2$$

at $t = t' \Rightarrow S(t-t'=0) = Z^{-1} \sum_{m,n} e^{-\beta E_m} \left|\langle m|A|n\rangle\right|^2$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{J(\omega)}{e^{\beta\omega}-1}\,d\omega = Z^{-1}\sum_{m,n} e^{-\beta E_m}\langle m|A|n\rangle\langle n|A|m\rangle \underbrace{\int \frac{(e^{\beta(E_m-E_n)}-1)\,\delta(\omega-(E_m-E_n))}{e^{\beta\omega}-1}}_{d\omega}$$

$$= Z^{-1}\sum_{m,n} e^{-\beta E_m}|\langle m|A|n\rangle|^2$$

$$\Rightarrow \quad S(t-t'=0) = \langle A^2\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{J(\omega)}{e^{\beta\omega}-1}\,d\omega.$$

From the expression

$$J(\omega) = 2\pi Z^{-1}\sum_{m,n} e^{-\beta E_m}|\langle m|A|n\rangle|^2 (e^{\beta(E_m-E_n)}-1)\,\delta(\omega-(E_m-E_n))$$

$$\text{if } \omega>0 \Rightarrow e^{\beta(E_m-E_n)}-1>0 \Rightarrow J(\omega)>0.$$

$$\omega = E_m - E_n$$

$$J(-\omega) = 2\pi Z^{-1}\sum_{m,n} e^{-\beta E_m}|\langle m|A|n\rangle|^2 (e^{\beta(E_m-E_n)}-1)\,\delta(\omega+(E_m-E_n))$$

$$= 2\pi Z^{-1}\sum_{m,n} e^{-\beta E_n}|\langle n|A|m\rangle|^2 (e^{\beta(E_n-E_m)}-1)\,\delta(\omega+(E_n-E_m))$$

$$= +\,2\pi Z^{-1}\sum_{m,n} e^{-\beta E_m}|\langle m|A|n\rangle|^2 (1-e^{\beta(E_m-E_n)})\,\delta(\omega-(E_m-E_n))$$

$$= -J(\omega)$$

1.3 $$\chi(\omega) = Z^{-1} \sum_{m,n} e^{-\beta E_m} |\langle m| x |n\rangle|^2 \frac{e^{\beta(E_m-E_n)}-1}{\hbar\omega - (E_m-E_n)+i\eta}$$

where $|n\rangle, |m\rangle$ are m, n's eigenstate of harmonic oscillator.

$$x = \sqrt{\frac{\hbar}{m\omega}} \frac{1}{\sqrt{2}} (a+a^\dagger) \Rightarrow |\langle m| a+a^\dagger |n\rangle|^2 = \delta_{m,n+1}\, m + \delta_{m,n-1}\, n$$

$$\Rightarrow \chi(\omega) = Z^{-1} \frac{\hbar}{2m\omega_0^2} \sum_{m,n} \frac{e^{-\beta E_n} - e^{-\beta E_m}}{\hbar\omega - (E_m-E_n)+i\eta} [\delta_{m,n+1}\, m + \delta_{m,n-1}\, n]$$

$$= Z^{-1} \frac{1}{2m\omega_0^2} \left[\sum_m \left\{ \frac{e^{-\beta E_m}(e^{\beta\omega} - 1)m}{\omega - \omega_0 + i\eta} \right\} + \sum_n \left\{ \frac{n e^{-\beta E_n}(1-e^{\beta\omega})}{\omega+\omega_0+i\eta} \right\} \right]$$

$$\frac{\sum_m m e^{-\beta E_m}}{Z} = \frac{\sum m e^{-\beta E_m}}{\sum e^{-\beta E_m}} = \frac{\sum m e^{-\beta m\hbar\omega}}{\sum e^{-\beta\hbar\omega}} = \frac{1}{e^{\beta\hbar\omega}-1}$$

$$\Rightarrow \chi(\omega) = \frac{1}{e^{\beta\hbar\omega}-1} \frac{1}{2m\omega_0^2} \left[\frac{e^{\beta\omega}-1}{\omega-\omega_0+i\eta} + \frac{1-e^{\beta\omega}}{\omega+\omega_0+i\eta} \right]$$

$$= \frac{1}{2m\omega_0} \left[\frac{1}{\omega-\omega_0+i\eta} - \frac{1}{\omega+\omega_0+i\eta} \right]$$

$$= \frac{1}{2m\omega_0} \frac{2\omega_0}{\omega^2-\omega_0^2+i\eta} = \frac{1}{m\ (\omega^2-\omega_0^2)+i\eta}$$

which agree with classic result.

$$\chi(\omega) = \frac{1}{m}\left[\frac{1}{\omega^2-\omega_0^2+i\eta}\right] \Rightarrow J(\omega) = -2\,\mathrm{Im}\,\chi(\omega) = \frac{2\pi}{m}\,\delta(\omega^2-\omega_0^2)$$

$$= \frac{2\pi}{2m\omega_0}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]$$

The pole of $\chi(\omega)$ is located at $\omega = \pm\omega_0$.

$$\langle x^2\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{J(\omega)}{e^{\beta\omega}-1}\,d\omega = \frac{1}{2m\omega_0}\left[\frac{1}{e^{\beta\omega_0}-1} - \frac{1}{e^{-\beta\omega_0}-1}\right]$$

when $\beta \to 0$

$$e^{\pm\beta\omega_0}-1 = \pm\beta\omega_0$$

$$\langle x^2\rangle = \frac{1}{2m\omega_0}\,2(\beta\omega_0)^{-1} = \frac{kT}{m\omega_0^2}$$

2. a $\quad H = H_0 + H_{int}$. for inter-acting electron-gas

$$H_{int} = \int \frac{\rho(r)\rho(r')\,dr}{|r-r'|} \quad \text{and} \quad \rho_q = \sum_q e^{iqr}\rho(r), \quad \text{thus}$$

$[\rho_q, H_{int}] = 0$. we only need to calculate $[[H_0, \rho_q]\ \rho_{-q}]$

$$[H_0\ \rho_q] = \sum_{kk'} \varepsilon_k [C_k^+ C_k,\ C_{k'-q}^+ C_{k'}] = \sum_{k} \varepsilon_k \left[C_k^+ C_{k+q} - C_{k-q}^+ C_k\right]$$

$$= \sum_k (\varepsilon_k - \varepsilon_{k+q})\ C_k^+ C_{k+q}$$

$$[[H_0, \rho_q]\ \rho_{-q}] = \sum_{kk'} (\varepsilon_k - \varepsilon_{k+q})\ [C_k^+ C_{k+q},\ C_{k'+q}^+ C_{k'}]$$

$$= \sum_k (\varepsilon_k - \varepsilon_{k+q})\ C_k^+ C_k - (\varepsilon_k - \varepsilon_{k+q})\ C_{k+q}^+ C_{k+q}$$

$$= \sum_k (\varepsilon_k - \varepsilon_{k+q} - \varepsilon_{k-q} + \varepsilon_k)\ C_k^+ C_k = -\sum_k [\varepsilon_{k+q} + \varepsilon_{k-q} - 2\varepsilon_k]\ C_k^+ C_k$$

$$\varepsilon_{k+q} + \varepsilon_{k-q} - 2\varepsilon_k = \left[k^2 + 2kq + q^2 + k^2 - 2kq + q^2 - 2k^2\right]/2m = \frac{\hbar^2 q^2}{m}$$

$$\Rightarrow [[H_0, \rho_q]\ \rho_{-q}] = -\frac{\hbar^2 q^2}{m} \sum_k C_k^+ C_k = -\frac{\hbar^2 q^2}{m} N$$

2.b. $$\chi(q,t) = \mp\frac{i}{\hbar}\,\Theta(t)\ \langle|\ \rho[q,t]\ \ \rho(-q,0)|\rangle \cdot \frac{1}{V}$$

Follow the reasoning in Prob 1 $\Rightarrow$

$$Im\,\chi(q,\omega) = \mp\frac{Z^{-1}}{V}(\pi) \sum_{m,n} e^{-\beta E_m} |\langle m|\rho_q|n\rangle|^2 (e^{\beta\omega}-1)\ \delta(\omega - (E_m - E_n))$$

$$\int_0^\infty d\omega\ \omega\ Im\,\chi(q,\omega) = -\frac{Z^{-1}\pi}{2V} \sum_{m,n} (e^{-\beta E_m} - e^{-\beta E_n})(E_m - E_n)\ |\langle m|\rho_q|n\rangle|^2$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} d\omega\ \omega\ Im\,\chi(q\,\omega)$$

$$\langle\uparrow[[H, \rho_q]\,\rho_{-q}]|\rangle = \frac{1}{Z}\sum_m e^{-\beta E_m}\Big\{\langle m|[H,\rho_q]|n\rangle\langle n|\rho_{-q}|m\rangle - \langle m|\rho_{-q}|n\rangle\langle n|[H,\rho_q]|m\rangle\Big\}$$

$$= \frac{1}{Z}\sum_m e^{-\beta E_m}\Big\{(E_m-E_n)\langle m|\rho_q|n\rangle\langle n|\rho_{-q}|m\rangle - \langle m|\rho_{-q}|n\rangle\langle n|\rho_q|m\rangle(E_n-E_m)\Big\}$$

$$= \frac{1}{Z}\sum_m \left(e^{-\beta E_m} - e^{-\beta E_n}\right)(E_m-E_n)\left(\langle m|\rho_q|n\rangle\langle n|\rho_{-q}|m\rangle\right)$$

$$\{[H, \rho_q]\,\rho_{-q}] = -\frac{Nq^2}{m}$$

$$\Rightarrow \int_0^\infty d\omega\, \omega\, \mathrm{Im}\,\chi(q,\omega) = \frac{Nq^2}{Vm}\cdot\frac{\pi}{2}$$

$\hookrightarrow$ f - sum rule.

More: Similarly.

$$\int_{-\infty}^{+\infty} d\omega\, \frac{\mathrm{Im}\,\chi(q,\omega)}{\omega} = -\frac{\pi}{ZV}\sum_{n,m} e^{-\beta E_m}\left(\frac{|\langle n|\rho_q^+|m\rangle|^2}{E_n-E_m} - \frac{|\langle n|\rho_q|m\rangle|^2}{E_m-E_n}\right)$$

which is just the real part of $\chi(q, \omega=0)$

$$\Rightarrow \int_0^{+\infty} d\omega\, \frac{\mathrm{Im}\,\chi(q,\omega)}{\omega} = \frac{1}{2}\int_{-\infty}^{+\infty} d\omega\, \frac{\mathrm{Im}\,\chi(q,\omega)}{\omega} = \frac{\pi}{2}\,\mathrm{Re}\,\chi(q,\omega=0)$$

$$\lim_{q\to 0} \frac{2}{\pi}\int_0^{+\infty} d\omega\, \frac{\mathrm{Im}\,\chi(q,\omega)}{\omega} = \mathrm{Re}\,\chi(q\to 0, \omega\to 0) = \frac{\partial n}{\partial \mu}.$$

($\omega \to 0$ first
$q \to 0$ second)

3 : 3a) the zero of dielectric function $\varepsilon(q.\omega)$ determines the excitation (plasmon spectrum)

$$\varepsilon(q,\omega) = 1 + \frac{4\pi e^2}{q^2} N(0) \int \frac{d\Omega}{4\pi} \frac{-\cos\theta}{s - \cos\theta + i\eta} \qquad (s = \frac{\omega}{v_F q})$$

at $q \to 0$.

at $s \gg 1$ $\quad \frac{-\cos\theta}{s-\cos\theta} = \frac{-\cos\theta/s}{1-\cos\theta/s} = -\frac{\cos\theta}{s}\left(1 + \frac{\cos\theta}{s} + \left(\frac{\cos\theta}{s}\right)^2 + \left(\frac{\cos\theta}{s}\right)^3\right]$

$$\Rightarrow \varepsilon(q.\omega) = 1 + \frac{4\pi e^2}{q^2} N(0) \left(\frac{-1}{3s^2} - \frac{1}{5s^4} \right)$$

keep to $1/s^2$ order, we have $\quad 1 = \frac{4\pi e^2 N(0)\ v_F^2}{3\ \omega^2} \Rightarrow \omega^2 = \omega_p^2$

keep to $1/s^4$ order

$$\varepsilon(q.\omega) = 1 - \left(\frac{\omega_p^2}{\omega^2} + \frac{3}{5} \frac{\omega_p^2}{\omega^4} (v_F q)^2 \right] = 0$$

$$\frac{\omega_p^2}{\omega^2} = 1 - \frac{3}{5}\left(\frac{\omega_p}{\omega}\right)^2 \left(\frac{v_F q}{\omega}\right)^2 \Rightarrow \frac{\omega^2}{\omega_p^2} \approx 1 + \frac{3}{5}\left(\frac{\omega_p}{\omega}\right)^2 \left(\frac{v_F q}{\omega}\right)^2$$

$$\approx 1 + \frac{3}{5}\left(\frac{v_F q}{\omega_p}\right)^2$$

3b) $$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{v}) = 0$$

$$\frac{\partial^2}{\partial t^2} n + \nabla \cdot \frac{\partial}{\partial t}(n\vec{v}) = 0 \, ,$$

from $$m\frac{\partial}{\partial t}(n\vec{v}) + m\vec{v}\cdot\nabla(n\vec{v}) = -ne\vec{E}$$

$$\nabla\cdot\frac{\partial}{\partial t}(n\vec{v}) + \nabla\left(\vec{v}\cdot\nabla(n\vec{v})\right) = -\frac{\nabla}{m}(ne\vec{E})$$

$$\Rightarrow \frac{\partial^2}{\partial t^2} n = \left(\nabla(\vec{v}\cdot\nabla(n\vec{v})\right) + \frac{\nabla}{m}(ne\vec{E})$$

$\nabla(\vec{v}\cdot\nabla(n\vec{v}))$ gives corrections at the order of $k^2 \rightarrow$ neglect eel.

$$\frac{\partial^2}{\partial t^2}(n_0 + \delta n) = \frac{\nabla}{m}(e(n+\delta n)\vec{E}) = \frac{e}{m}\nabla(n+\delta n)\vec{E} + e(n+\delta n)\frac{\nabla\vec{E}}{m}$$

$$\nabla\vec{E} = -4\pi e\,\delta n \quad \Rightarrow \quad \vec{E}\propto\delta n\text{, keep to linear order}$$

$$\frac{\partial^2}{\partial t^2}\delta n = -\frac{4\pi e^2}{m} n_0\,\delta n \quad \Rightarrow \quad \omega_p^2 = \frac{4\pi n_0 e^2}{m}$$

4): a) $$\delta \mathcal{E}_{HF}(k) = -\frac{1}{V}\sum_q V_q\, n_{k+q} = -\int \frac{d^3q}{(2\pi)^3}\, \frac{4\pi e^2}{q^2 + k_{TF}^2}\, n_{k+q}$$

$$k_{TF} = 4\pi e^2 \frac{\partial n}{\partial \mu} .$$

$$\left(\frac{k_{TF}}{k_f}\right)^2 = \frac{4}{\pi}\frac{1}{k_f a_0} , \qquad \text{define} \quad \frac{4}{3}\pi r_s^3 = 1/n \Rightarrow k_f = \left(\frac{9}{4}\pi\right)^{1/3} r_s^{-1}$$

$$\Rightarrow \left(\frac{k_{TF}}{k_f}\right)^2 = \frac{4}{\pi}\left(\frac{4}{9\pi}\right)^{1/3}\frac{r_s}{a_0} \approx 0.7\,\frac{r_s}{a_0}$$

in the dilute limit $q \propto k_F \propto \frac{1}{r_s}$, $k_{TF} \propto r_s^{-1/2} \Rightarrow k_{TF} \gg k_F$

we can neglect $q^2 \Rightarrow q^2 + k_{TF}^2 \approx k_{TF}^2$

$$\Rightarrow \delta \mathcal{E}_{HF}(k) = -\frac{1}{V}\frac{4\pi e^2}{k_{TF}^2}\sum_q n_{k+q} = -\frac{n}{2}\frac{4\pi e^2}{k_{TF}^2}$$

↑ only sum over particles with the same spin.

b) with a finite polarization

$$\delta \mathcal{E}_{HF\uparrow}(k) = -\frac{1}{V}\frac{4\pi e^2}{k_{TF}^2}\sum_q n_{k+q\uparrow} = -\frac{n}{2}(1+p)\frac{4\pi e^2}{k_{TF}^2(p)}$$

$$\delta \mathcal{E}_{HF\downarrow}(k) = -\frac{n}{2}(1-p)\frac{4\pi e^2}{k_{TF}^2(p)}$$

$k_{TF}(p)$ is the the T-F screening wavevector with polarization p

$$k_{TF} = 4\pi e^2\left[\left(\frac{\partial n}{\partial \mu}\right)_\uparrow + \left(\frac{\partial n}{\partial \mu}\right)_\downarrow\right] \Rightarrow \frac{k_{TF}(p)}{k_{TF}} = \frac{\left(\frac{\partial n}{\partial \mu}\right)_\uparrow + \left(\frac{\partial n}{\partial \mu}\right)_\downarrow}{\left(\frac{\partial n}{\partial \mu}\right)(\text{at } p=0)}$$

we know $\frac{\partial n}{\partial \mu} \propto \frac{4\pi k_F^2}{v_F} \propto k_F$ in 3D $\Rightarrow$

$$k_{TF}(p)/k_{TF} = \left[(1+p)^{1/3} + (1-p)^{1/3}\right]/2 = 1 - 1/9\, p^2$$

$$\Rightarrow \delta \mathcal{E}_{HF\uparrow}(k) = -\frac{n}{2} \frac{1+p}{1-1/9\, p^2} \frac{4\pi e^2}{k_{TF}^2(p=0)}, \quad \delta \mathcal{E}_{HF\downarrow}(k) = -\frac{n}{2} \frac{1-p}{1-1/9\, p^2} \frac{4\pi e^2}{k_{TF}^2(p=0)}$$

at small p.

The total HF energy

$$E_{ex\uparrow} = \frac{N_\uparrow}{2} \delta \mathcal{E}_{HF}(k_\uparrow) = -\frac{V}{2} n_\uparrow^2 \frac{4\pi e^2}{k_{TF}^2(p)}, \quad E_{ex\downarrow} = -\frac{V}{2} n_\downarrow^2 \frac{4\pi e^2}{k_{TF}^2(p=0)}$$

$$\Rightarrow \frac{E_{ex}}{V} = -\frac{1}{2} \frac{1}{4} n^2 \left[(1+p)^2 + (1-p)^2\right] (1 - 1/9\, p^2)^{-1} \frac{4\pi e^2}{k_{TF}^2(p=0)}$$

$$= -\frac{n^2}{4} (1 + \frac{10}{9} p^2) \cdot \frac{4\pi e^2}{k_{TF}^2(p=0)}$$

The kinetic energy with Polarization p should be.

$$E_{K\uparrow} = \frac{3}{5} N_\uparrow \mathcal{E}_{f\uparrow}^0 = \frac{3}{5} \frac{1+p}{2} N_0\, \mathcal{E}_f^0 (1+p)^{\frac{2}{3}}$$

$$E_{K\downarrow} = \frac{3}{5} \frac{1-p}{2} N\, \mathcal{E}_f^0 (1-p)^{2/3}$$

$$\Rightarrow E_K = \frac{3}{5} N \mathcal{E}_f^0 \left[(1+p)^{5/3} + (1-p)^{5/3}\right] = \frac{3}{5} N \mathcal{E}_f^0 (1 + 5/9\, p^2]$$

$$\Rightarrow \frac{E_{tot}}{V} = \frac{3}{5} n\, \mathcal{E}_f^0 (1 + \frac{5}{9} p^2) - \frac{n^2}{4} (1 + \frac{10}{9} p^2) \frac{4\pi e^2}{k_{TF}^2(p=0)}.$$

d) $$\frac{dE}{V} = B\frac{dM}{V} = \mu_B B\, n\, dp$$

$$\mu_B B\cdot n = \frac{\partial E/V}{\partial p} = \frac{2}{3} n \mathcal{E}_f^0 p - \frac{5}{9} n^2 p \cdot \frac{4\pi e^2}{k_{TF}^2(p=0)}$$

$$\chi_{HF} = \frac{M}{B} = \frac{n\mu_B p}{B} = \frac{n^2\mu_B^2 p}{\mu_B B\cdot n} = \frac{n^2\mu_B^2 p}{\frac{2}{3} n\mathcal{E}_f^0 p - \frac{5}{9} n^2 p\frac{4\pi e^2}{k_{TF}^2(p=0)}}$$

$$= \frac{n\mu_B^2}{\frac{2}{3}\mathcal{E}_f^0 - \frac{5}{9} n\frac{4\pi e^2}{k_{TF}^2(p=0)}}$$

For free fermion $\Rightarrow$ $\chi_0 = \dfrac{n\mu_B^2}{\frac{2}{3}\mathcal{E}_f^0}$

$$\Rightarrow \frac{\chi_{HF}}{\chi} = \frac{\frac{2}{3}\mathcal{E}_f^0}{\frac{2}{3}\mathcal{E}_f^0 - \frac{5}{9} n\frac{4\pi e^2}{k_{TF}^2(p=0)}} = \frac{1}{1 - \frac{5}{6}\frac{n}{\mathcal{E}_f^0}\frac{4\pi e^2}{k_{TF}^2}}$$

$$\frac{n}{\mathcal{E}_f}\cdot\frac{4\pi e^2}{k_{TF}^2} = \frac{n}{\mathcal{E}_f^0}\cdot\frac{1}{\frac{\partial n}{\partial \mu}} = \frac{n}{\mathcal{E}_f^0}\cdot\frac{1}{\frac{3}{2}\frac{n}{\mathcal{E}_f^0}} = \frac{2}{3}$$

$$\frac{\chi_{HF}}{\chi} = \frac{1}{1-5/6\cdot\frac{2}{3}} \simeq \frac{1}{4/9} = 2.25$$

whether ferromagnetism can occur in ~~a~~ electron gas is a subtle issue, and we will neglect it.