PHYS 212A: Homework 1

October 28, 2013

1.1

a

To solve this we use the completeness relations to expand $\langle\alpha|$ in the a' basis and $\langle\beta|$ in the a'' basis. Thus we have,

$$|\alpha\rangle\langle\beta| = \sum_{a'}\sum_{a''} |a'\rangle\langle a'|\alpha\rangle\langle\beta|a''\rangle\langle a''|\,. \tag{1}$$

From this expression, we can read off the $i, j-th$ matrix element

$$[|\alpha\rangle\langle\beta|]_{ij} = \langle a_i|\alpha\rangle\langle a_j|\beta\rangle^* \tag{2}$$

b

Setting $\langle\alpha| = \langle s_z = \hbar/2|$ and $\langle\beta| = \langle s_x = \hbar/2|$, we can write

$$|\alpha\rangle\langle\beta| = \begin{pmatrix} \langle+|+\rangle\langle+|s_x = \hbar/2\rangle^* & \langle+|+\rangle\langle-|s_x = \hbar/2\rangle^* \\ \langle-|+\rangle\langle+|s_x = \hbar/2\rangle^* & \langle-|+\rangle\langle-|s_x = \hbar/2\rangle^* \end{pmatrix} = 1/\sqrt{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{3}$$

1.8

To prove these identities, we should consider all posible combinations of spin operators under the commutator and anti-commutator. An example of one case will be given as the other cases proceed similarly.

$$\begin{gathered} [S_x, S_z] \\ = S_x S_z - S_z S_x \\ = \frac{\hbar^2}{4}\left[(|+\rangle\langle-| + |-\rangle\langle+|)(|+\rangle\langle+| - |-\rangle\langle-|) - (|+\rangle\langle+| - |-\rangle\langle-|)(|+\rangle\langle-| + |-\rangle\langle+|)\right] \\ = \frac{\hbar^2}{2}(-|+\rangle\langle-| + |-\rangle\langle+|) \\ = -i\hbar S_y \end{gathered}$$

1.9

The strategy for this problem is to first express the operator in Cartesian coordinates, and then find eigenstates of the operator in the usual fashion. Based on the diagram provided we can write the Cartesian expansion of $\vec{S} \cdot \vec{n}$,

$$\vec{S} \cdot \vec{n} = \sin(\beta)\cos(\alpha)S_x + \sin(\beta)\sin(\alpha)S_y + \cos(\beta)S_z \tag{4}$$

With the standard s_z representation of the S operators, we can now write the eigenvalue equation,

$$\begin{pmatrix} \cos(\beta) & \sin(\beta)\cos(\alpha) - i\sin(\beta)\sin(\alpha) \\ \sin(\beta)\cos(\alpha) + i\sin(\beta)\sin(\alpha) & -\cos(\beta) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Additionally, we have the normalization condition,

$$|a|^2 + |b|^2 = 1. \tag{5}$$

Solving one of the lines of the matrix equation along with the normalization condition yields $a = \cos(\frac{\beta}{2})$ and $b = e^{i\alpha}\sin(\frac{\beta}{2})$.

1.13

We will represent the total measurement operator as the product of the three measurements described in the problem. Taking account of the order in which they should act on an incoming state we can write,

$$M_T = M_- M_n M_+, \tag{6}$$

where M_n gives the measurement along an arbitary direction as described in Problem 1.9, with $\alpha = 0$. Expanding each of these operators in the s_z basis, we have

$$M_T = (|-\rangle\langle-|)(\cos(\beta/2)|+\rangle + \sin(\beta/2)|-\rangle)(\cos(\beta/2)\langle+| + \sin(\beta/2)\langle-|)((|+\rangle\langle+|)) \tag{7}$$

Multiplying all the terms out and taking advantage of the orthogonality relations, we find

$$M_T = |-\rangle \sin(\beta/2)\cos(\beta/2)\langle+| \tag{8}$$

With the result of the first meaurement normalized to unity, we have that the probablity of getting a $|+\rangle$ state in and a $|-\rangle$ state out is given by

$$|\langle-| M_T |+\rangle|^2 = \sin^2(\beta/2)\cos^2(\beta/2) = \sin^2(\beta)/4 \tag{9}$$

The maximum of this function occurs at $\pi/2$ at which point the probability of the measurement is 1/4.