PHYS 212A: Homework 3

November 11, 2013

Exercise 2

a

Take $\hat{n}$ along the z-axis and $r_i = r_x$. The idea is to reduce the commutator to position and momentum operators so that we can take advantage of the cannonical quantization conditions. Plugging in to eq (13) we find,

$$[\hat{n} \cdot \vec{J}, x] = [L_z, x] + [S_z, x] = [xp_y - yp_x, x] = x(xp_y - yp_x) - (xp_y - yp_x)x = x[x, p_y] + y[p_x, x] \quad (1)$$

We can now apply the cannonical quantization conditions to find

$$i\alpha[J_z, x] = y = -i\alpha(i\hbar)y \quad (2)$$

which implies $\alpha = 1/\hbar$.

b

Choose $\hat{n} = \hat{j}$, then for infinitesimal rotations we can neglect terms of $O(\theta^2)$ and have

$$D^\dagger(g)S_iD(g) = (1 + \frac{i\theta}{\hbar}(J_j))S_i(1 - \frac{i\theta}{\hbar}(J_j)) = S_i + \frac{i\theta}{\hbar}(S_jS_i - S_iS_j) = S_i + \frac{i\theta}{\hbar}[S_j, S_i] \quad (3)$$

Comparing this to the relation

$$D^\dagger(g)S_iD(g) = g_{ij}S_k, \quad (4)$$

we can see that we must have

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k \quad (5)$$

c

This derivation follows from the same procedure as used above.

Exercise 4

Note: For an alternative derivation of this result see Sakurai 2.7

Taking the partial derivative with respect to time of the ψ', we have

$$\frac{\partial \psi'}{\partial t} = e^{\frac{ie}{\hbar c}f}\frac{\partial \psi}{\partial t} + e^{\frac{ie}{\hbar c}f}\frac{ie}{hc}\frac{\partial f}{\partial t} = e^{\frac{ie}{\hbar c}f}(\frac{\partial \psi}{\partial t} + \frac{ie}{\hbar}(\phi - \phi')) \quad (6)$$

Rearranging the above expression, we find

$$(i\hbar\frac{\partial}{\partial t} - e\phi')\psi' = e^{\frac{ie}{\hbar c}f}(i\hbar\frac{\partial}{\partial t} - e\phi)\psi \tag{7}$$

Acting on the new wavefunction with the spatial gradient operator leads to the expression

$$(-i\hbar\nabla - \frac{e}{c}A')\psi' = e^{\frac{ie}{\hbar c}f}(-i\hbar\nabla - \frac{e}{c}A)\psi \tag{8}$$

Combining these two results shows that the new wavefunction satisfies

$$i\hbar\frac{\partial}{\partial t}\psi' = H'\psi' \tag{9}$$

2.3

a

For this problem the Hamiltonian is simply

$$H = -\vec{\mu}\cdot\vec{B} = (g_S\mu_B/2)\sigma_z B \tag{10}$$

Recall the eigenstates of the operator $\vec{S}\cdot\hat{n}$ from problem 1.9. The normalized eigenket is

$$|\psi\rangle = (\frac{1+\cos\beta}{2})^{1/2}\begin{pmatrix} 1 \\ \frac{\sin\beta}{\cos\beta+1} \end{pmatrix} \tag{11}$$

From the Schrodinger equation, we have

$$-i\omega\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \partial/\partial t\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} \tag{12}$$

Solving the Schrodinger equation using the using the normalized eigenket of $\vec{S}\cdot\hat{n}$, we find that the the time evolutoin of the wavefunction is described by

$$\psi(t) = \begin{pmatrix} (\frac{1+\cos\beta}{2})^{1/2}e^{-i\omega t} \\ (\frac{\sin\beta}{(2(1+\cos\beta))^{1/2})}e^{i\omega t} \end{pmatrix} \tag{13}$$

If we now change to the s_x basis the coefficients we find that the coefficient of $|s_x;+\rangle$ is

$$a_1 = 1/2^{1/2}(\frac{1+\cos\beta}{2})^{1/2}e^{-i\omega t} + 1/2^{1/2}(\frac{\sin\beta}{(2(1+\cos\beta))^{1/2})}e^{i\omega t} \tag{14}$$

To find the probability of measuring the electron in the $|s_x;+\rangle$ we calculate

$$a_1^*a_1 = 1/2(1+\sin\beta\cos 2\omega t) \tag{15}$$

b

The expectation value is given by

$$\langle s_x\rangle = \langle\psi(t)|s_x|\psi(t)\rangle = (A^*(t), B^*(t))\hbar/2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \hbar/2\sin\beta\cos 2\omega t \tag{16}$$

c

As $\beta \to 0$ the probability of measuring $s_x = \hbar/2 \to 1/2$ and $\langle s_x \rangle = 0$. In the other limit $s_x = \hbar/2 \to 1/2(1 + \cos 2\omega t$ and $\langle s_x \rangle = \hbar(\cos^2 \omega t - 1/2)$.

2.11

For a one-dimensional SHO potential $H = p^2/2m + \frac{1}{2} m\omega^2 x^2$, hence $\dot{x} = (1/i\hbar)[x,H] = p/m$, and $\dot{p} = (1/i\hbar)[p,H] = (1/i\hbar)(m\omega^2/2)[p,x^2] = (m\omega^2/2i\hbar)[-2i\hbar x] = -m\omega^2 x$. Hence $\ddot{x} + \omega^2 x = 0$, and solution is $x(t) = A\cos\omega t + B\sin\omega t$. At $t=0$, $x(0) = A$ while $\dot{x}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$ leads to $\dot{x}(0) = B\omega$ and thus $p(0) = m\omega B$. Thus in the Heisenberg picture $x(t) = x(0)\cos\omega t + (p(0)/m\omega)\sin\omega t$.

Our state vector $|\alpha\rangle = e^{-ipa/\hbar}|0\rangle$ at $t=0$; for $t>0$ we have in the Heisenberg picture $\langle x(t)\rangle = \langle\alpha|x(t)|\alpha\rangle$. We note that

$$e^{ip(0)a/\hbar}x(0)e^{-ip(0)a/\hbar} = e^{ip(0)a/\hbar}([x(0),e^{-ip(0)a/\hbar}] + e^{-ip(0)a/\hbar}x(0))$$
$$= x(0) + a,$$

while $e^{ip(0)a/\hbar}p(0)e^{-ip(0)a/\hbar} = p(0)$. Hence

$$\langle x(t)\rangle = \langle\alpha|x(t)|\alpha\rangle = \langle 0|e^{ipa/\hbar}x(t)e^{-ipa/\hbar}|0\rangle$$
$$= \langle 0|e^{ip(0)a/\hbar}[x(0)\cos\omega t+(p(0)/m\omega)\sin\omega t]e^{-ipa/\hbar}|0\rangle.$$

Since $\langle 0|x(0)|0\rangle = \langle 0|p(0)|0\rangle = 0$, we obtain for $\langle x(t)\rangle = a\cos\omega t$.

2.19

(a) Take $a|\lambda\rangle = \exp[-|\lambda|^2/2]\ a\ \exp[\lambda a^\dagger]\ |0\rangle = \exp[-|\lambda|^2/2]\ a \sum_{n=0}^{\infty}(\lambda^n/n!)(a^\dagger)^n|0\rangle$;
but we know that $(a^\dagger)^k|n\rangle = \sqrt{(n+1)(n+2).....(n+k)}\,|n+k\rangle$ hence $(a^\dagger)^k|0\rangle = \sqrt{k!}\,|k\rangle$
and $a(a^\dagger)^k|0\rangle = \sqrt{k!}\ a|k\rangle = \sqrt{k}\sqrt{k!}\,|k-1\rangle$. Thus $a|\lambda\rangle = e^{-|\lambda|^2/2}\sum_{n=1}^{\infty}\lambda^n\frac{\sqrt{n}\sqrt{n!}}{n!}|n-1\rangle =$

$= e^{-|\lambda|^2/2}\sum_{n=0}^{\infty}\lambda^{n+1}(\sqrt{n+1}/\sqrt{(n+1)!})\,|n\rangle$. But $(n+1)!/(n+1) = n!$, hence

$$a|\lambda\rangle = e^{-|\lambda|^2/2}\sum_{n=0}^{\infty}(\lambda^{n+1}/\sqrt{n!})\,|n\rangle = \lambda e^{-|\lambda|^2/2}\sum_{n=0}^{\infty}(\lambda^n/\sqrt{n!})\,|n\rangle. \qquad (1)$$

The r.h.s. of (1) is $\lambda e^{-|\lambda|^2/2}e^{\lambda a^\dagger}|0\rangle$ by noting that $e^{\lambda a^\dagger}|0\rangle = \sum_{n=0}^{\infty}(\lambda a^\dagger)^n/n!\ |0\rangle$
$= \sum_{n=0}^{\infty}\lambda^n|n\rangle/\sqrt{n!}$. Hence with $|\lambda\rangle = e^{-|\lambda|^2/2}e^{\lambda a^\dagger}|0\rangle$, we have indeed $a|\lambda\rangle = \lambda|\lambda\rangle$ with
λ in general a complex number. For normalization we find

$$\begin{aligned}\langle\lambda|\lambda\rangle &= e^{-|\lambda|^2}\langle 0|e^{\lambda^* a}e^{\lambda a^\dagger}|0\rangle = e^{-|\lambda|^2}\langle 0|e^{\lambda^* a}\sum_{n=0}^{\infty}\lambda^n|n\rangle/\sqrt{n!} \\ &= e^{-|\lambda|^2}\langle 0|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(\lambda^n/\sqrt{n!})(\lambda^* a)^m/m!\ |n\rangle\ ,\end{aligned} \qquad (2)$$

but $a^m|n\rangle = \sqrt{n(n-1).....(n-m+1)}\,|n-m\rangle$, hence (2) contributes by orthonormality of states only when $n-m = 0$, i.e.

$$\langle\lambda|\lambda\rangle = e^{-|\lambda|^2}\langle 0| \sum_{n=0}^{\infty} \frac{\lambda^n(\lambda^*)^n}{\sqrt{n!}(n!)} \sqrt{n!}|0\rangle = e^{-|\lambda|^2}e^{+|\lambda|^2} = 1.$$

Therefore $|\lambda\rangle$ is a normalized coherent state.

(b) $\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$, $x = \sqrt{\hbar/2m\omega}(a+a^\dagger)$, where $a|\lambda\rangle = \lambda|\lambda\rangle$ and $\langle\lambda|a^\dagger = \langle\lambda|\lambda^*$. So $\langle x\rangle = \langle\lambda|x|\lambda\rangle = \sqrt{\hbar/2m\omega}(\langle\lambda|(a+a^\dagger)|\lambda\rangle) = \sqrt{\hbar/2m\omega}(\lambda+\lambda^*)$, and $\langle x\rangle^2 = (\hbar/2m\omega)(\lambda^2+\lambda^{*2}+2\lambda\lambda^*) = (\hbar/2m\omega)(\lambda+\lambda^*)^2$. Now $x^2 = xx = (\hbar/2m\omega)[a^{\dagger 2}+a^2+aa^\dagger+a^\dagger a] = (\hbar/2m\omega)[a^{\dagger 2}+a^2+2a^\dagger a+1]$, hence $\langle x^2\rangle = (\hbar/2m\omega)[\lambda^{*2}+\lambda^2+2\lambda^*\lambda+1] = (\hbar/2m\omega)[(\lambda^*+\lambda)^2+1]$. Likewise $\langle p\rangle^2 = -(\hbar m\omega/2)[\lambda^*-\lambda]^2$ and $\langle p^2\rangle = (\hbar m\omega/2)[1-(\lambda^*-\lambda)^2]$, using $p = i\sqrt{\hbar m\omega/2}(a^\dagger-a)$. Hence $\langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2 = \hbar m\omega/2$ and $\langle(\Delta x)^2\rangle = \hbar/2m\omega$ and $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \hbar^2/4$.

(c) Write $|\lambda\rangle = e^{-|\lambda|^2/2}\sum_{n=0}^{\infty}(\lambda^n/\sqrt{n!})|n\rangle = \sum_{n=0}^{\infty} f(n)|n\rangle$, hence $f(n) = e^{-|\lambda|^2/2}(\frac{\lambda^n}{\sqrt{n!}})$. Therefore $|f(n)|^2 = e^{-|\lambda|^2}|\lambda|^{2n}/n!$ and is a Poisson distribution

$$P(\lambda',n) = e^{-\lambda'}\lambda'^n/n!, \text{ where } \lambda' = |\lambda|^2.$$

Now $\Gamma(n+1) = n!$, hence $|f(n)|^2 = e^{-|\lambda|^2}|\lambda|^{2n}/\Gamma(n+1)$. The maximum value is obtained by noting that $\ln|f(n)|^2 = -|\lambda|^2 + n\ln[|\lambda|^2] - \ln\Gamma(n+1)$, and $\frac{\partial}{\partial n}\ln|f(n)|^2 = \ln|\lambda|^2 - \frac{\partial}{\partial n}\ln\Gamma(n+1) = 0$. The latter equation defines n_{max} where for large n, $\frac{\partial}{\partial n}\ln\Gamma(n+1) \sim \ln n$. Hence $n_{max} = |\lambda|^2$.

(d) The translation operator $e^{-ip\ell/\hbar}$ where p is momentum operator and ℓ just the displacement distance, can be rewritten as

$$e^{-ip\ell/\hbar} = e^{\ell\sqrt{m\omega/2\hbar}(a^\dagger-a)} = e^{\ell\sqrt{m\omega/2\hbar}a^\dagger - \ell\sqrt{m\omega/2\hbar}a}\, e^{-\frac{1}{2}(-\ell^2)(m\omega/2\hbar)[a^\dagger,a]}$$

$$= e^{-\frac{1}{2}(-\ell^2)(m\omega/2\hbar)[a^\dagger,a]}e^{\ell\sqrt{m\omega/2\hbar}a^\dagger}e^{-\ell\sqrt{m\omega/2\hbar}a} = e^{\frac{-\ell^2 m\omega}{4\hbar}}\, e^{\ell\sqrt{m\omega/2\hbar}a^\dagger}\, e^{-\ell\sqrt{m\omega/2\hbar}a}.$$

Note $e^{-\ell\sqrt{m\omega/2\hbar}a}|0\rangle = |0\rangle$ because $a|0\rangle = 0$. Hence

$$e^{-ip\ell/\hbar}|0\rangle = e^{-|\lambda|^2/2}\, e^{\lambda a^\dagger}|0\rangle, \text{ where } \lambda = \ell\sqrt{m\omega/2\hbar}$$

[We have used here the identity $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$, true for any pair of operators A and B that commute with [A,B], c.f. R. J. Glauber, Phys. Rev. <u>84</u>, 399 (1951).]