PHYS 212A: Homework 5

December 8, 2013

3.21

a

We define $L_i = \epsilon_{ijk} x_j p_k$. Writing x and p in terms of creation and annihilation operators gives,

$$L_i = \epsilon_{ijk} i\hbar (a_j^\dagger + a_j)(a_k^\dagger - a_k) = \epsilon_{ijk} \frac{i\hbar}{2}(a_j a_k^\dagger - a_j^\dagger a_k) = \epsilon_{ijk} i\hbar a_j a_k^\dagger \tag{1}$$

The other result can be derived similarly.

b

The states $ketqlm$ are defined as eigenstates of the L_z and L^2 operators. Here since $N = 2q + l = n_x + n_y + n_z = 1$, we have three possible kets $|100\rangle, |010\rangle, |001\rangle$. It is helpful to consider what happens to each state under the angular momentum operators.

$$L_z |100\rangle = i\hbar(a_x a_y^\dagger - a_y a_x^\dagger) |100\rangle = i\hbar |010\rangle \tag{2}$$
$$L_z |010\rangle = -i\hbar |100\rangle \tag{3}$$
$$L_z |001\rangle = 0 \tag{4}$$

Under L^2 any of the 3 states will have the same result, $L^2 |001\rangle = \hbar^2[1(1+1)] |001\rangle$, this shows $|010\rangle_{qlm} = |001\rangle_n$. From inspection, we can determine that $|01 \pm 1\rangle_{qlm} = \frac{1}{\sqrt{2}}(|100\rangle_n \pm i |010\rangle_n)$

c

Now we have 6 possible states to consider, and find that

$$|020\rangle_{qlm} = \frac{1}{\sqrt{3}}(|200\rangle + |020\rangle + |002\rangle) \tag{5}$$

d

Following the same procedure as above, we find

$$\begin{aligned}
L_z |200\rangle &= i\hbar(|110\rangle) && (6)\\
L_z |020\rangle &= -i\hbar(|110\rangle) && (7)\\
L_z |110\rangle &= i\hbar\sqrt{2}(|020\rangle - |200\rangle) && (8)\\
L_z |101\rangle &= i\hbar(|011\rangle) && (9)\\
L_z |011\rangle &= -i\hbar(|101\rangle) && (10)\\
& && (11)
\end{aligned}$$

By inspection, we can determine the unnormalized $m = \pm 1$ states are $|101\rangle \pm i\,|011\rangle$. Similarly, for $m = \pm 2$ states are $\frac{1}{\sqrt{2}}(|200\rangle - |020\rangle) \pm i\,|110\rangle$

3.22

a

For $x = 0$ we have

$$g(x,t) = 1/(1-t) = 1 + t + t^2 + ... = \sum L_n(0)\frac{t^n}{n!} \tag{12}$$

Comparing powers of t, we can see $L_n(0) = \frac{\partial^n g(x,t)}{\partial t^n}|_{t=0} = n!$ The other result follows similarly.

b

$$\frac{\partial g}{\partial x} = \frac{-t}{1-t} g \tag{13}$$

Inserting the series expression gives

$$(t-1)\sum L_n'(x)\frac{t^n}{n!} - \sum L_n'(x)\frac{t^n}{n!} = t\sum L_n(x)\frac{t^n}{n!} \tag{14}$$

Rearranging indices to collect like powers of t, we find

$$nL_{n-1}'(x) - L_n'(x) = nL_{n+1}(x) \tag{15}$$

c

$$\frac{\partial g}{\partial t} = \frac{1-t-x}{(1-t)^2} g \tag{16}$$

$$(1-2t+t^2)\sum L_n(x)\frac{t^{n-1}}{(n-1)!} = (1-t-x)\sum L_n(x)\frac{t^n}{n!} \tag{17}$$

Again grouping powers of t, we find the sought after expression.

d

This is most easily proved using the generator

$$xg'' + (1-x)g' + ng = 0 - \frac{e^{\frac{-tx}{1-t}} t(1-x)}{(1-t)^2} + \frac{e^{\frac{-tx}{1-t}} t^2 x}{(1-t)^3} + t[\frac{e^{\frac{-tx}{1-t}}}{(1-t)^2} - \frac{e^{\frac{-tx}{1-t}} x}{(1-t)^2} - \frac{e^{\frac{-tx}{1-t}} tx}{(1-t)^3}] = 0 \quad (18)$$

3.15

3.24

3.26

$\cdots \; 2m \; \overline{rR} \; \overline{dr^2}(rR)$

(3)

From $L_\pm = L_x \pm iL_y$, we have $L_x = \frac{1}{2}(L_+ + L_-)$ and $L_y = \frac{-i}{2}(L_+ - L_-)$, and from (3.5.39) and (3.5.40) $L_\pm|\ell,m\rangle = c_\pm(\ell,m)|\ell,m\pm1\rangle = \hbar[\ell(\ell+1)-m(m\pm1)]^{1/2}|\ell,m\pm1\rangle$. Hence $\langle L_x\rangle = \langle\ell m|\frac{1}{2}(L_+ + L_-)|\ell m\rangle = 0$ since $\langle\ell m|\ell m'\rangle = \delta_{mm'}$. Similarly $\langle L_y\rangle = \langle\ell m|L_y|\ell m\rangle = 0$. Now $\langle L_x^2\rangle = \langle\ell m|\frac{1}{4}(L_+L_+ + L_+L_- + L_-L_+ + L_-L_-)|\ell m\rangle$. But $L_+L_-|\ell m\rangle = c_-(\ell,m) \times c_+(\ell,m-1)|\ell m\rangle$ and $L_-L_+|\ell m\rangle = c_+(\ell,m)c_-(\ell,m+1)|\ell m\rangle$ while $\langle\ell m|L_+L_+|\ell m\rangle = \langle\ell m|L_-L_-|\ell m\rangle = 0$ since states of different m values are orthogonal. Hence $\langle L_x^2\rangle = \frac{1}{4}\langle\ell m|L_+L_- + L_-L_+|\ell m\rangle = \frac{1}{4}\{c_-(\ell,m)c_+(\ell,m-1) + c_+(\ell,m)c_-(\ell,m+1)\} = \frac{1}{4}\{c_-^2(\ell,m) + c_+^2(\ell,m)\} = \frac{\hbar^2}{4}\{\ell(\ell+1)-m(m-1)+\ell(\ell+1)-m(m+1)\} = \frac{\hbar^2}{2}\{\ell(\ell+1)-m^2\}$. Similarly $\langle L_y^2\rangle = \langle\ell m|-\frac{1}{4}(L_+L_+ - L_+L_- - L_-L_+ + L_-L_-)|\ell m\rangle = \frac{1}{4}\langle\ell m|(L_+L_- + L_-L_+)|\ell m\rangle = \langle L_x^2\rangle$.

<u>Semiclassical interpretation</u>: We know that $\vec{L}^2|\ell m\rangle = \hbar^2\ell(\ell+1)|\ell m\rangle$, $L_z^2|\ell m\rangle = \hbar^2 m^2 |\ell m\rangle$. Thus $\langle\vec{L}^2\rangle = \ell(\ell+1)\hbar^2$ and $\langle L_z^2\rangle = m^2\hbar^2$. In the classical correspondence $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2$ expresses itself in terms of the corresponding expectation values, and indeed $\langle L_x^2\rangle + \langle L_y^2\rangle + \langle L_z^2\rangle = \frac{1}{2}\hbar^2(\ell(\ell+1)-m^2) + \frac{1}{2}\hbar^2(\ell(\ell+1)-m^2) + m^2\hbar^2 = \ell(\ell+1)\hbar^2 = \langle\vec{L}^2\rangle$.

We are to add angular momenta $j_1 = 1$ and $j_2 = 1$ to form $j = 2,1,0$ states. Express all nine $\{j,m\}$ eigenkets in terms of $|j_1j_2,m_1m_2\rangle$. The simplest states are $j_1=1, m_1=\pm1$; $j_2=1, m_2=\pm1$, i.e. $|j=2,m=2\rangle = |{+}{+}\rangle$ and likewise $|j=2,m=-2\rangle = |{-}{-}\rangle$. Using the ladder operator method we have $J_- = J_{1-} \oplus J_{2-}$ and (setting $\hbar = 1$ for convenience) from (3.5.40) $J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}|j,m-1\rangle$. So $J_-|j=2,m=2\rangle = \sqrt{4}|j=2,m=1\rangle = (J_{1-} \oplus J_{2-})|j_1=1,j_2=1; m_1=1,m_2=1\rangle = \sqrt{2}|0{+}\rangle + \sqrt{2}|{+}0\rangle$, i.e.

$|j=2,m=1\rangle = \frac{1}{2^{1/2}}(|0+\rangle + |+0\rangle)$. Now $J_-|j=2,m=1\rangle = \sqrt{6}|j=2,m=0\rangle = (J_{1-} \oplus J_{2-}) \times [\frac{1}{2^{1/2}}(|0+\rangle + |+0\rangle)] = |-+\rangle + 2|00\rangle + |+-\rangle$. Hence $|j=2,m=0\rangle = \frac{1}{6^{1/2}}(|-+\rangle+2|00\rangle+|+-\rangle)$. Also $J_-|j=2,m=0\rangle= \sqrt{6}|j=2,m=-1\rangle= \frac{1}{6^{1/2}}(\sqrt{2}|-0\rangle+2\sqrt{2}|0-\rangle+2\sqrt{2}|-0\rangle+\sqrt{2}|0-\rangle)$, therefore $|j=2,m=-1\rangle = \frac{1}{2^{1/2}}(|-0\rangle+|0-\rangle)$.

For the j=1 states, let us recognize that $|11\rangle = a|0+\rangle + b|+0\rangle$ with normalization $|a|^2 + |b|^2 = 1$. Since $\langle 21|11\rangle = 0$ by orthogonality, we have $a+b = 0$. Choosing our phase convention to be real, we can write $|11\rangle = \frac{1}{2^{1/2}}(|+0\rangle - |0+\rangle)$. Applying next $J_- = J_{1-} \oplus J_{2-}$ to the two sides respectively, we have $|10\rangle = \frac{1}{2^{1/2}}(|+-\rangle - |-+\rangle)$ and similarly $|1-1\rangle = \frac{1}{2^{1/2}}(|0-\rangle - |-0\rangle)$.

Finally we may write $|j=0,m=0\rangle = \alpha|+-\rangle + \beta|00\rangle + \gamma|-+\rangle$, determine α,β,γ by normalization $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$ and orthogonality to $|j=1,m=0\rangle$ and $|j=2,m=0\rangle$. Choosing α,β,γ to be real we have $|j=0,m=0\rangle = \frac{1}{3^{1/2}}(|+-\rangle - |00\rangle + |-+\rangle)$.

(a) We have $J_y = \frac{1}{2i}(J_+ - J_-)$, then using (3.5.41) we derive easily

$$\langle jm'|J_y|jm\rangle = \frac{\hbar}{2i}[\sqrt{j(j+1)-m(m+1)}\langle jm'|j,m+1\rangle-\sqrt{j(j+1)-m(m-1)}\langle jm'|j,m-1\rangle]$$

and therefore for m and m' = +1,0,−1 and j=1 one finds the matrix form for $\langle j=1,m'|J_y|j=1,m\rangle$ as depicted in (3.5.54).

(b) Unlike the $j=\frac{1}{2}$ case, for j=1 only $[J_y^{(j=1)}]^2$ is <u>independent</u> of <u>1</u> and $J_y^{(j=1)}$, and in fact we have $(J_y/\hbar)^{2m+1} = (J_y/\hbar)$ and $(J_y/\hbar)^{2n} = (J_y/\hbar)^2$ where m and n are positive integers. By expansion of the exponential $e^{-iJ_y\beta/\hbar}$ in power series

$$e^{-iJ_y\beta/\hbar} = \sum_{n=0}^{\infty} \frac{(-iJ_y\beta/\hbar)^{2n}}{(2n)!} + \sum_{m=0}^{\infty} \frac{(-iJ_y\beta/\hbar)^{2m+1}}{(2m+1)!}$$

$$= \underline{1} + (J_y/\hbar)^2 \sum_{n=1}^{\infty} \frac{(+\beta)^{2n}(-1)^n}{(2n)!} - i(J_y/\hbar) \sum_{m=0}^{\infty} \frac{(+\beta)^{2m+1}(-1)^m}{(2m+1)!}$$

$$= \underline{1} - (J_y/\hbar)^2(1-\cos\beta) - i(J_y/\hbar)\sin\beta .$$

(c) Insert the 3×3 matrix form for J_y from (a), i.e. (3.5.54), into the exponential of part (b) above, we find

$$d^{(j=1)}(\beta) = e^{-iJ_y\beta/\hbar} = \begin{pmatrix} \frac{1+\cos\beta}{2} & -\sin\beta/\sqrt{2} & \frac{1-\cos\beta}{2} \\ \sin\beta/\sqrt{2} & \cos\beta & -\sin\beta/\sqrt{2} \\ \frac{1-\cos\beta}{2} & \sin\beta/\sqrt{2} & \frac{1+\cos\beta}{2} \end{pmatrix}$$

which is (3.5.57).