Home Work #4

Problem 1. (Sakurai 2nd edition 6.4)

Solution:

We quote the formulas in Sakurai's book for logarithmic derivative and phase shift:

$$\beta_\ell = \left(\frac{r}{j_\ell(k'r)}\, j'_\ell(k'r)\right)\Big|_{r=R}$$

$$\tan\delta_\ell = \frac{kR\, j'_\ell(kR) - \beta_\ell\, j_\ell(kR)}{kR\, n'_\ell(kR) - \beta_\ell\, n_\ell(kR)}$$

in which $\frac{\hbar^2}{2m}k'^2 = E - V_0$, appearing in the logarithmic derivative at $R-0$, while the expression for $\tan\delta_\ell$ is a matching of connecting condition between $R-0$ and $R+0$.

Since we're considering low energy scattering (i.e. $kR \ll 1$), only a small number of partial wave channels contribute. And we can only keep the results to lowest non-vanishing order of kR, which would be $(kR)^{2\ell+1}$ for ℓ'th partial wave channel known as the so-called threshold behaviour.

The expansions for $j_\ell(x)$ and $n_\ell(x)$ around $x=0$ are:

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}}\, J_{\ell+1/2}(x) = \frac{\sqrt{\pi}}{2}\sum_{n=0}^{+\infty}\frac{(-)^n}{n!\,\Gamma(n+\ell+3/2)}\left(\frac{x}{2}\right)^{2n+\ell}$$

$$n_\ell(x) = \sqrt{\frac{\pi}{2x}}\, N_{\ell+1/2}(x) = (-)^{\ell+1}\frac{\sqrt{\pi}}{2}\sum_{n=0}^{+\infty}\frac{(-)^n}{n!\,\Gamma(n-\ell+1/2)}\left(\frac{x}{2}\right)^{2n-\ell-1}$$

These give

$$j_\ell(x) = \frac{x^\ell}{(2\ell+1)!!} - \frac{x^{\ell+2}}{2(2\ell+3)!!} + O(x^{\ell+4})$$

$$j'_\ell(x) = \frac{\ell x^{\ell-1}}{(2\ell+1)!!} - \frac{(\ell+2)x^{\ell+1}}{2(2\ell+3)!!} + O(x^{\ell+3})$$

$$n_\ell(x) = -\frac{(2\ell-1)!!}{x^{\ell+1}} - \frac{(2\ell-3)!!}{2x^{\ell-1}} + O\left(\frac{1}{x^{\ell-3}}\right)$$

$$n'_\ell(x) = \frac{(\ell+1)(2\ell-1)!!}{x^{\ell+2}} + \frac{(\ell-1)(2\ell-3)!!}{2x^{\ell}} + O\left(\frac{1}{x^{\ell-2}}\right)$$

Denoting $x' = k'R$, $x = kR$, we have

$$\beta_\ell = x'\frac{j'_\ell(x')}{j_\ell(x')} = \ell - \frac{1}{2\ell+3}x'^2 + O(x'^4)$$

Numerator for $\tan\delta_\ell$ is

$$x\, j'_\ell(x) - \beta_\ell(x')\, j_\ell(x) = \frac{\ell x^\ell}{(2\ell+1)!!} - \frac{(\ell+2)x^{\ell+2}}{2(2\ell+3)!!} - \left(\ell - \frac{\ell+1}{2(2\ell+3)}x'^2\right)\left(\frac{x^\ell}{(2\ell+1)!!} - \frac{x^{\ell+2}}{2(2\ell+3)!!}\right) + O(x^{\ell+4})$$

$$= \frac{x^{\ell+2}}{(2\ell+3)!!}\left(\left(\frac{x'}{x}\right)^2 - 1\right) + O(x^{\ell+4})$$

Denominator is

$$x n_\ell'(x) - \beta_\ell(x') n_\ell(x) = x \cdot \frac{(\ell+1)(2\ell-1)!!}{x^{\ell+2}} + \ell \cdot \frac{(2\ell-1)!!}{x^{\ell+1}} + O\left(\frac{1}{x^{\ell-1}}\right)$$

$$= \frac{(2\ell+1)!!}{x^{\ell+1}} + O\left(\frac{1}{x^{\ell-1}}\right)$$

Thus

$$\tan\delta_\ell = \frac{1}{(2\ell+1)!!(2\ell+3)!!} x^{2\ell+3} \left(\left(\frac{x'}{x}\right)^2 - 1 \right) + \text{higher order terms}$$

Since

$$\left(\frac{x'}{x}\right)^2 - 1 = \frac{E-V_0}{E} - 1 = -\frac{V_0}{E} = -\frac{2mV_0R^2}{\hbar^2(kR)^2}.$$

We have

$$\tan\delta_\ell = -\frac{2mV_0R^2}{\hbar^2} \frac{1}{(2\ell+1)!!(2\ell+3)!!} (kR)^{2\ell+1} + \text{higher order terms}.$$

Thus $\delta_\ell \simeq \tan\delta_\ell \simeq \sin\delta_\ell \simeq -\frac{2mV_0R^2}{\hbar^2} \frac{1}{(2\ell+1)!!(2\ell+3)!!} (kR)^{2\ell+1}.$

$$\sigma_0 = 4\pi \frac{d\sigma_0}{d\Omega} = 4\pi |f_0|^2 \simeq 4\pi \frac{1}{k^2} \delta_0^2 \simeq \frac{16\pi}{9} \frac{m^2V_0^2R^6}{\hbar^4}.$$

So for small enough k, the total cross section can be taken as that of s-wave channel, and is $\frac{16\pi}{9}\frac{m^2V_0^2R^6}{\hbar^4}$.

If we keep the result up to p-wave, then

$$f(\theta) = \frac{1}{k}(\delta_0 + 3\delta_1 \cos\theta)$$

$$= \frac{1}{k}\delta_0 \left(1 + \frac{1}{5}(kR)^2\cos\theta\right)$$

Then

$$\frac{d\sigma}{d\theta} = |f(\theta)|^2 = f_0 \left(1 + \frac{1}{5}(kR)^2\cos\theta\right)$$

$$\simeq f_0\left(1 + \frac{2}{5}(kR)^2\cos\theta\right)$$

Hence $B/A = \frac{2}{5}(kR)^2.$

Problem 2. Sakurai 2nd Edition 6.5.

Solution:

(a) In a spherically symmetric scattering potential, the partial wave expansion for scattering amplitude $f(\theta)$ is

$$f(\theta) = \frac{1}{k}\sum_{l=0}(2l+1)\,e^{i\delta_l}\sin\delta_l\,P_l(\cos\theta)$$
$$\simeq \sum_{l=0}(2l+1)\frac{\delta_l}{k}P_l(\cos\theta)\quad,\ \text{for small enough } \delta_l.$$

Use the orthogonality relation for Legendre polynomials

$$\int_{-1}^{1}dx\,P_l(x)\,P_{l'}(x) = \frac{2}{2l+1}\,\delta_{ll'}.$$

We get

$$(2l+1)\frac{\delta_l}{k}\cdot\frac{2}{2l+1} \simeq \int_0^{\pi}\sin\theta\,d\theta\,f(\theta)\,P_l(\cos\theta)$$
$$= \int_{-1}^{1}dx\,P_l(x)\cdot(-)\frac{2mV_0}{\hbar^2\mu}\,\frac{1}{2k^2(1-x)+\mu^2}$$
$$= \int_{-1}^{1}dx\cdot P_l(x)\cdot(-)\frac{mV_0}{\hbar^2\mu k^2}\,\frac{1}{1+\frac{\mu^2}{2k^2}-x}$$
$$= -\frac{2mV_0}{\hbar^2\mu k^2}\,\frac{1}{2}\int_{-1}^{1}dx\cdot\frac{P_l(x)}{1+\frac{\mu^2}{2k^2}-x}$$
$$= -\frac{2mV_0}{\hbar^2\mu k^2}\,Q_l\left(1+\frac{\mu^2}{2k^2}\right)$$

$$\Rightarrow \delta_l = \frac{-mV_0}{\hbar^2\mu k}\,Q_l\left(1+\frac{\mu^2}{2k^2}\right)$$

(b)(i) Obviously $1+\frac{\mu^2}{2k^2} > 1$, and so the expansion of $Q_l(\xi)$ applies, particularly $Q_l(\xi) > 0$. So $\delta_l > 0$, $V_0 < 0$ and $\delta_l < 0$ if $V_0 > 0$.

(ii) When $\frac{1}{k} \gg \frac{1}{\mu}$. i.e. $\frac{\mu}{k} \gg 1$, we have

$$Q_l\left(1+\frac{\mu^2}{2k^2}\right) \simeq Q_l\left(\frac{\mu^2}{2k^2}\right)$$
$$\simeq \frac{l!}{(2l+1)!!}\,\frac{1}{\left(\frac{\mu^2}{2k^2}\right)^{l+1}}$$
$$= \frac{2^{l+1}\,l!}{(2l+1)!!}\,\mu^{-2(l+1)}\cdot k^{2l+2}$$

$$\Rightarrow \delta_l = -\frac{2^{l+1}\,l!}{(2l+1)!!}\,\frac{mV_0}{\hbar^2\mu^{2l+3}}\,k^{2l+1}$$

Problem 3. Sakurai 2nd Edition 6.10.

Solution:

(a) Radial equation for s-wave is

$$\frac{d^2u}{dr^2}+\left(k^2-\frac{2m}{\hbar^2}V(r)\right)u=0$$

In this problem, we have

$$\frac{d^2u}{dr^2}+\left(k^2-\gamma\delta(r-R)\right)u=0.$$

For $r<R$, we have $\frac{d^2u}{dr^2}+k^2u=0$.

Imposing the boundary condition $u|_{r=0}=0$, one has

$$u(r)=A\sin kr.$$

For $r>R$, we have $u(r)=B\sin(kr+\delta_0)$.

From the differential equation, we have

$$\int_{R-}^{R+}\frac{d^2u}{dr^2}\,dr=\int_{R-}^{R+}\left(\gamma\delta(r-R)-k^2\right)u(r)\,dr$$

$$=\gamma u(R)$$

Thus the connecting conditions at $r=R$ are

$$\begin{cases} u(r=R-)=u(r=R+) \\ \left.\frac{du}{dr}\right|_{R+}-\left.\frac{du}{dr}\right|_{R-}=\gamma u(R) \end{cases}$$

$$\Rightarrow\quad \left.\frac{1}{u}\frac{du}{dr}\right|_{R+}-\left.\frac{1}{u}\frac{du}{dr}\right|_{R-}=\gamma.$$

For $r<R$, $\frac{1}{u}\frac{du}{dr}=k\cot kr$; for $r>R$, $\frac{1}{u}\frac{du}{dr}=k\cot(kr+\delta_0)$.

Hence

$$k\cot(kR+\delta_0)-k\cot kR=\gamma$$

$$\Rightarrow\quad \tan\delta_0=-\frac{\frac{\gamma}{k}\tan kR}{\frac{\gamma}{k}+\tan kR+\cot kR}=-\frac{\frac{\gamma}{k}\sin^2 kR}{1+\frac{\gamma}{k}\sin kR\cos kR}$$

So the equation for phase shift is

$$\tan\delta_0=-\frac{\frac{\gamma}{k}\sin^2 kR}{1+\frac{\gamma}{k}\sin kR\cos kR}$$

(b) ① (If $\tan kR$ is not close to zero, one gets hard sphere scattering.)

Using $\gamma\gg k$, i.e. $\frac{\gamma}{k}\gg 1$, we get

$$\tan\delta_0\simeq-\frac{\frac{\gamma}{k}\sin^2 kR}{\frac{\gamma}{k}\sin kR\cos kR}=-\tan kR,$$

which is just the hard sphere result.

② (Resonance)

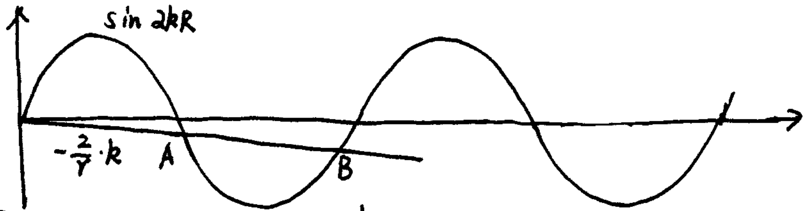
Resonance occurs when cross section for the partial wave channel reaches its maximal value while at the same time $\cot\delta_0$ goes through zero from positive side as k increases.

$$\cot\delta_0 = -\frac{1+\frac{\gamma}{k}\sin kR\cos kR}{\frac{\gamma}{k}\sin^2 kR}$$

$$= -\frac{1+\frac{\gamma}{2k}\sin 2kR}{\frac{\gamma}{k}\sin^2 kR}$$

$$= -\frac{1}{2}\,\frac{\sin 2kR + \frac{2k}{\gamma}}{\sin^2 kR} = -\frac{1}{2}\,\frac{\sin 2kR - \left(-\frac{2k}{\gamma}\right)}{\sin^2 kR}$$

Let $\cot\delta_0 = 0$, we have $\sin 2kR = -\frac{2k}{\gamma} \to 0$, $\gamma \to +\infty$. So for large γ, $\sin 2kR$ is very close to 0. If we require $\cot\delta_0$ to pass through zero from positive side then $\sin kR - \left(-\frac{2k}{\gamma}\right)$ has to pass from negative side.



The slope $\left|-\frac{2}{\gamma}\right| \ll 2R$ since $\gamma \gg \frac{1}{R}$. From graph we see that B is true solution while A is not, so $2kR = 2n\pi + x$, where x is small.

Then ~~2kR~~

$$\sin(2kR) = \sin x = -\frac{2k}{\gamma} \Rightarrow x \simeq -\frac{2k}{\gamma}.$$

Hence $2kR = 2n\pi - \frac{2k}{\gamma}$, or $kR = n\pi - \frac{k}{\gamma}$.

③ (Determine position of resonance to order $\frac{1}{\gamma}$; compare result with spherical well)

Resonance position has been determined in ② as $kR = n\pi - \frac{k}{\gamma}$,

or $kR = \frac{n\pi}{1+\frac{1}{\gamma R}} \simeq n\pi\left(1-\frac{1}{\gamma R}\right)$

For the quantum well, let $\sin kR = n\pi$, we get $kR = n\pi$.

One can see that positions of resonance is quite close to bound states in a quantum well in the limit of large γ.

④ (Obtain an expression for resonance width)

$$\frac{d\cot\delta_0}{dE} = \frac{dk}{dE}\,\frac{d\cot\delta_0}{dk}$$

$$= \frac{1}{\frac{\hbar^2}{m}k}\cdot\frac{d}{dk}\left((-)\,\frac{1+\frac{\gamma}{k}\sin kR\cos kR}{\frac{\gamma}{k}\sin^2 kR}\right)$$

$$= -\frac{m}{\hbar^2 k}\,\frac{d}{dk}\left(\frac{k+\gamma\sin kR\cos kR}{\gamma\sin^2 kR}\right)$$

$$= -\frac{m}{\hbar^2 k}\cdot\frac{1}{\gamma^2\sin^4 kR}\Big[\gamma\sin^2 kR\left(1+\gamma R\left(\cos^2 kR - \sin^2 kR\right)\right) - (k+\gamma\sin kR\cos kR)\cdot\gamma R\,2\sin kR\cdot\cos kR\Big]$$

$$= -\frac{m}{\hbar^2 k}\;\frac{1}{\gamma \sin^3 kR}\left[\sin kR\left[1+\gamma R(\cos^2 kR - \sin^2 kR)\right] - 2R\cos kR\,(k+\gamma \sin kR \cos kR)\right]$$

$\Rightarrow$ At $E=E_r$, i.e. $k=k_r=\frac{n\pi}{R}\left(1-\frac{1}{\gamma R}\right)$, we can replace $\sin kR$ with $\frac{n\pi}{\gamma R}(-1)^{n-1}$, $\cos kR$ with $(-1)^n$.

Then

$$\left.\frac{d\cot\delta_0}{dE}\right|_{E=E_r} = \frac{m}{\hbar^2 k}\;\frac{1}{\gamma\left(\frac{n\pi}{\gamma R}\right)^3\cdot(-1)^{3n-3}}\cdot 2kR\,(-1)^n\cdot$$

$$= -\frac{2mR}{\hbar^2\gamma\cdot\left(\frac{n\pi}{\gamma R}\right)^3}$$

$$\Rightarrow \Gamma = \frac{\hbar^2\gamma\cdot\left(\frac{n\pi}{\gamma R}\right)^3}{mR} = \frac{\hbar^2 n^3\pi^3}{mR^4\gamma^2} \propto \frac{1}{\gamma^2}$$

So Γ decreases as γ increases.