Lect 25 Zero energy scattering, bound states, resonances

§ Zero energy scattering (low energy limit — non-perturbative)

Let us consider a short-range scattering potential, the S-wave WF can be written as $\psi = \frac{1}{\sqrt{4\pi}} R_0(r) = \frac{u(r)}{r}$, where $u(r)$ satisfies

$$\frac{d^2}{dr^2} u + \left[k^2 - \frac{2m}{\hbar^2} V(r)\right] u = 0 \,, \quad \text{where } k = \sqrt{\frac{2mE}{\hbar^2}} \,.$$

Ex: please prove this Eq.

Consider the limit of $k \to 0$, at $r > R$, where R is the interaction range, we have $\frac{d^2}{dr^2} u = 0 \Rightarrow \boxed{u(r) = \text{const}\cdot\left(1 - \frac{r}{a_0}\right)}$

- if $a_0 = 0 \Rightarrow u(r) \propto r \propto \sin kr$ (no-phase shift, free space)

  $a_0 \to \pm\infty \Rightarrow u(r) \propto \text{const} \propto \sin(kr + \frac{\pi}{2})$ (maximum phase shift strong scattering!)

Now let us relate a_0 with phase shift δ_0 and scattering amplitude f_0.

at $r > R$, $u(r) \simeq A \sin(kr + \delta_0) = A\left[\sin\delta_0 + \cos\delta_0 \cdot kr\right]$

$$\simeq (1 + k\cot\delta_0 \, r)$$

$$\Rightarrow \quad \boxed{k\cot\delta_0(k)\Big|_{k=0} = -\frac{1}{a_0}}$$

If $k \neq 0$, but remains small, generally speaking, we expect $k\cot\delta_0(k)$ is an even function of k.

Ex: prove the above conclusion.

- we have the expansion $\boxed{k \cot \delta_0(k) \simeq -\frac{1}{a_0} + \frac{k^2}{2} R \quad \text{at } k \to 0}$

later we will see R is at the same order of inteaction range, and thus we do not use a new symbol.

The scattering amplitude $f_0 = \frac{\sqrt{4\pi}}{k} e^{i\delta_0} \sin\delta_0 = \frac{\sqrt{4\pi}}{k} \frac{1}{\cot\delta_0 - i}$

$$f_0 = \sqrt{4\pi} \frac{1}{k\cot\delta_0 - ik} = \boxed{\sqrt{4\pi} \frac{1}{\frac{-1}{a_0} - ik + \frac{k^2}{2}R} = f_0}$$

- If a_0 is finite $\Rightarrow$ $\delta_0 \simeq \tan\delta_0 \simeq -k a_0 \longleftarrow$ hard sphere with radius a_0

If $a_0 \to \pm\infty$, $\delta_0 = \pm\frac{\pi}{2}$, $\Rightarrow$ $f_0 = \frac{\sqrt{4\pi}\, i}{k}$

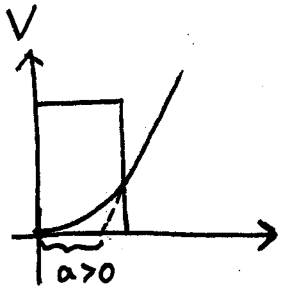
$$\Rightarrow \quad \sigma_0 = \frac{4\pi}{k^2} = \frac{2\pi\hbar^2}{mE} \propto \frac{1}{E} \longleftarrow \boxed{\text{resonance scattering}}$$

§ Evolution of δ_0 as varying interaction potential

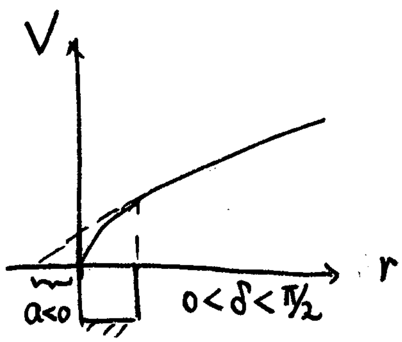
In the case of attractive interaction, we will see δ_0 is quite complicated. A series of resonance scatterings appear as varying the depth of potential well.

③

repulsive potential



attractive potential

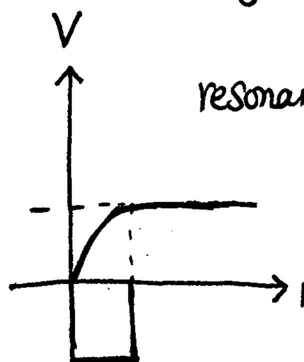


$0<\delta<\pi/2$

→

$\sigma = \frac{4\pi}{k^2}$

resonance scattering

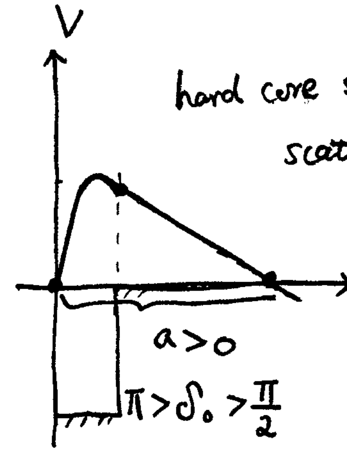


$a = \mp\infty$, $\delta_0 = \frac{\pi}{2} \mp 0^+$

→ ,

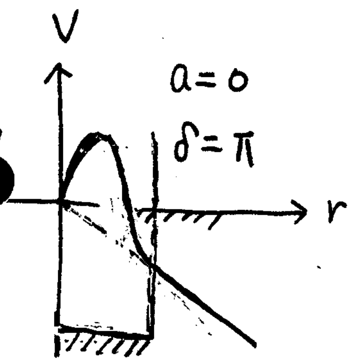
$\sigma = 4\pi a^2$

hard core sphere scattering



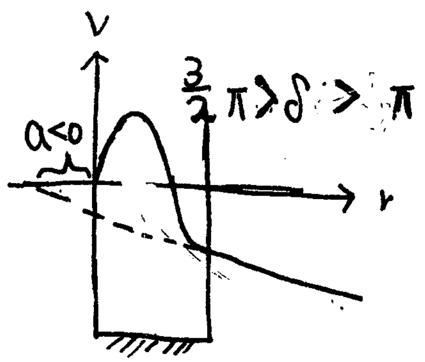
$\pi > \delta_0 > \frac{\pi}{2}$

→

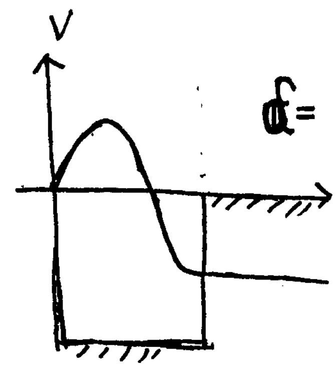


$a = 0$

$\delta = \pi$



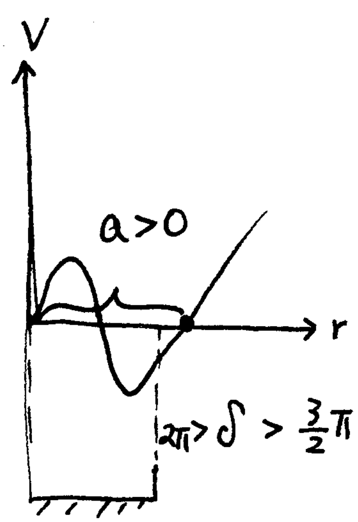
$\frac{3}{2}\pi > \delta > \pi$



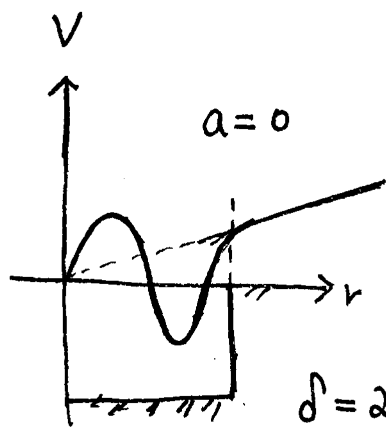
$\delta = \frac{3}{2}\pi$

$a = \mp\infty$

resonance scattering



$2\pi > \delta > \frac{3}{2}\pi$

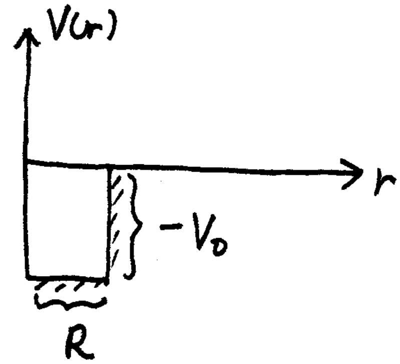


$a = 0$

$\delta = 2\pi$

§ Bound state:

The appearence of resonance is closely related to the bound state in the potential well.



Conside a spherical potential well problem and conside the case there's just a bound state at the top of the well.

$$\begin{cases} \dfrac{d^2 u_b}{dr^2} + k_0^2 u_b = 0 & (r<R) \qquad k_0 = \sqrt{\dfrac{2mV}{\hbar^2}} \\ \dfrac{d^2 u_b}{dr^2} - \beta^2 u_b = 0 & (r>R) \qquad \beta \to 0^+ , \end{cases} \qquad u_b = r\psi_b(r)$$

$$\left.\begin{array}{ll} \text{at } r<R, & u = \sin k_0 r \\ \phantom{\text{at }} r>R & u = A e^{-\beta r} \end{array}\right\} \Rightarrow \begin{array}{l} \left.\dfrac{u'}{u}\right|_{r=R} = k_0 \cot k_0 R \\ \left.\dfrac{u'}{u}\right|_{r=R} = -\beta e^{-\beta R} \end{array}$$

$$\Rightarrow \quad k_0 \cot k_0 R = -\beta e^{-\beta R} \to 0 \text{ as } \beta \to 0^+$$

thus $k_0 R \simeq (n+\frac{1}{2})\pi + 0^+$, let's denote $\Delta\delta = k_0 R - (n+\frac{1}{2})\pi$

$$k_0 R \cot k_0 R \simeq (n+\tfrac{1}{2})\pi \cot[\tfrac{\pi}{2} + \Delta\delta] = -(n+\tfrac{1}{2})\pi\,\Delta\delta$$

$$\Rightarrow \boxed{\beta R = (n+\tfrac{1}{2}\pi)\,\Delta\delta} \longleftarrow \text{localization length}$$

Now let us consider the zero energy scattering states.

$$\begin{cases} \dfrac{d^2 u_s}{dr^2} + k_0^2 u_s = 0 & (r<R) \qquad u_s = r\psi_s(r) \\ \dfrac{d^2 u_s}{dr^2} + k^2 u_s = 0 & (r>R) \qquad k = \sqrt{\dfrac{2mE}{\hbar^2}} \end{cases}$$

at $r<R$, $u_s \simeq u_b \simeq \sin k_0 r$

$r>R$ $\quad u_s \simeq A\sin(kr+\delta_0)$.

match boundary condition $\left.\frac{u_s'}{u_s}\right|_{r=R^+} = \left.\frac{u_s'}{u_s}\right|_{r=R^-}$

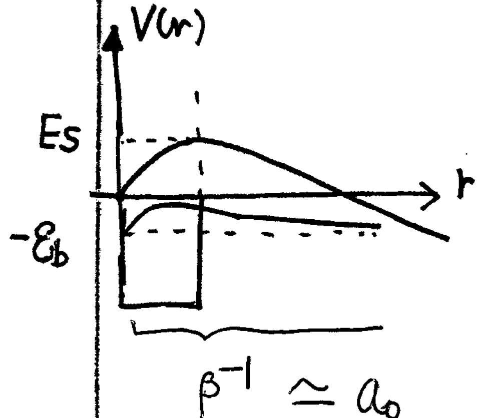
$$kR\cot(kR+\delta_0) = k_0 R\cot k_0 R = -\beta R\, e^{-\beta R} \quad \text{and} \begin{cases}\beta R\to 0\\ kR\to 0\end{cases}$$

$$\Rightarrow \quad kR\cot\delta_0 \simeq -\beta R \quad \Rightarrow \frac{-1}{a_0} = k\cot\delta_0 = -\beta \quad \Rightarrow a_0 = \beta^{-1} > 0$$

thus $$f_0 = \sqrt{4\pi}\,\frac{1}{k\cot\delta_0 - ik} = \frac{-\sqrt{4\pi}}{\beta+ik} = \frac{-\sqrt{4\pi}}{\beta + i\sqrt{\frac{2mE}{\hbar^2}}}$$

> Again the bound state $E_b = \frac{-\hbar^2\beta^2}{2m}$ can be obtained as the pole of $f_0(E)$ in the first Riemann sheet; i.e. $\sqrt{\frac{2mE_b}{\hbar^2}} = i\beta$.

⊛ When there is just a true bound state ~~[illegible]~~ below the top of the well, the scattering length $a_0>0$, which approximatly equals to the localization length of the shallow bound state β^{-1}.



as $\mathcal{E}_b \to 0$, $a_0 \to +\infty$

condition for the first resonance

$$\boxed{k_0 R = \frac{\pi}{2} + 0^+}$$

Scattering state v.s. bound state

⊛ If the well is made even shallower, $k_0R < \frac{\pi}{2}$, there will be no bound state. Denote $\Delta\delta = k_0R - \frac{\pi}{2} < 0 \Rightarrow$

$$kR\cot(kR+\delta_0) = k_0R\cot k_0R \simeq \frac{\pi}{2}\cot\left[\frac{\pi}{2}+\Delta\delta\right] = -\frac{\pi}{2}\Delta\delta$$

$$kR\cot\delta_0 = -\frac{\pi}{2}\Delta\delta \;\Rightarrow\; \frac{-1}{a_0} = k\cot\delta_0 = -\frac{\pi}{2R}\Delta\delta$$

or $\quad a_0 = \frac{2}{\pi}\frac{R}{\Delta\delta} < 0.$

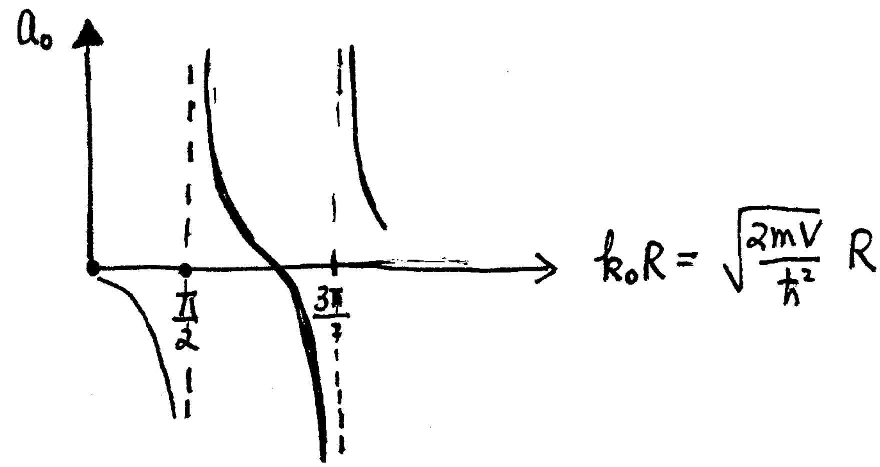
⊛ Combine $k_0R = \frac{\pi}{2} \pm 0^+$ cases together, we have

$$a_0 = \frac{}{\pi/2}\,\frac{R}{k_0R-\pi/2} \simeq \frac{1}{k_0}\,\frac{1}{k_0R-\pi/2} \qquad \text{where } k_0 = \sqrt{\frac{2mV}{\hbar^2}}$$

more generally, if $k_0R \simeq (n+\frac{1}{2})\pi$, we have

$$a_0 \simeq \frac{1}{k_0}\,\frac{1}{k_0R-(n+1/2)\pi}$$

As we deepen the potential well, more and more bound states are trapped, and a_0 oscilates.



⊛ Near the resonances, the cross section and scattering length have no depenedence on R, V_0 (microscopic parameters). This is the virtue of resonance scattering.

$$f_0 = \frac{\sqrt{4\pi}}{-\frac{1}{a_0} - ik} \Rightarrow \sigma = |f_0|^2 = \frac{4\pi\hbar^2}{2m(E_k + |\mathcal{E}_b|)} \xrightarrow[\mathcal{E}_b \to 0]{} \frac{4\pi\hbar^2}{2mE_k}$$

$$k\cot\delta_0 = \frac{-1}{a_0} \xrightarrow{a_0 \to \pm\infty} \delta_0 = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

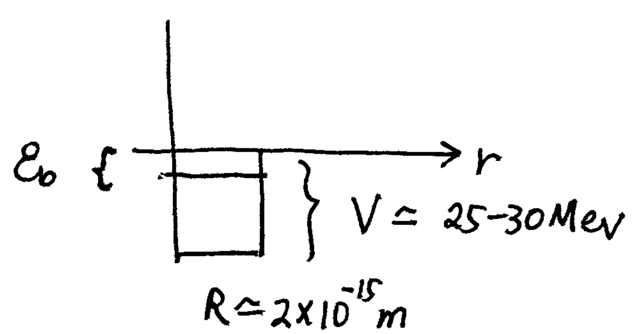
Ramsauer - Townsen effect

as $k_0 R \sim n\pi$, then the phase shift $\rightarrow \delta_0 \sim n\pi$,

scattering length $a_0 \to 0$, we get perfect transmission.

⊛ example of shallow bound state (nuclear physics).

hot neutron scattering on proton. There's a boundned ~~state~~ D with $\mathcal{E}_b = +2.23$ Mev.



hot neutron $E_k \simeq \frac{1}{40}$ eV

reduced mass $\mu \simeq m/2$

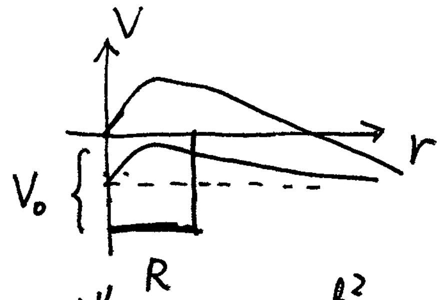
$$k = \sqrt{\frac{2\mu E}{\hbar}} = 1.7\times10^{10}\, m^{-1}$$

$kR \simeq 3.5\times10^{-5} \ll 1$, which the limit of zero energy scattering

a_0 measured $\simeq 5.4\times10^{-15}$ m $\Rightarrow \mathcal{E}_{b,\,\text{estimated}} \simeq \frac{\hbar^2}{2\mu a^2} = 1.4$ Mev.

(*) The relation between scattering length and bound state decay length

(to k^2 order)



For scattering state, the boundary condition at $r = R$ $\Rightarrow$

① $k \cot(kR + \delta_0) = k_1 \cot k_1 R$, where $k_1 = (k^2 + k_0^2)^{1/2} \simeq k_0 + \dfrac{k^2}{2k_0}$

for bound state $E_b = -\dfrac{\hbar^2\beta^2}{2m}$

② $k' \cot k'R = -\beta$, where $k' = \sqrt{k_0^2 - \beta^2} = k_0 - \dfrac{\beta^2}{2k_0}$

From ② $\left(k_0 - \dfrac{\beta^2}{2k_0}\right) \cot\left(k_0 R - \dfrac{\beta^2}{2k_0} R\right) = -\beta$

$$\cot\left(k_0 R - \frac{\beta^2}{2k_0} R\right) = -\frac{\beta}{k_0} \frac{1}{1 - \beta^2/2k_0^2} \simeq -\beta/k_0$$

$$\cot\left(\frac{\pi}{2} + k_0 R - \frac{\pi}{2} - \frac{\beta^2}{2k_0} R\right) = -\tan\left[k_0 R - \frac{\pi}{2} - \frac{\beta^2}{2k_0} R\right] \Rightarrow$$

$$k_0 R - \frac{\pi}{2} - \frac{\beta^2}{2k_0} R = \beta/k_0 \Rightarrow \boxed{k_0 R - \frac{\pi}{2} = \frac{\beta}{k_0} + \frac{\beta^2 R}{2k_0}}$$

From ①

$$k\cot(kR + \delta_0) = k \frac{1 - \tan kR \cot^{-1}\delta_0}{\tan kR + \cot^{-1}\delta_0} = k \frac{\cot\delta_0 - \tan kR}{1 + \cot\delta_0 \tan kR}$$

$$\simeq \frac{k\cot\delta_0 - k^2 R}{1 + k\cot\delta_0 R}$$

$$k_1 \cot k_1 R = \left(k_0 + \frac{k^2}{2k_0}\right) \cot\left[\frac{\pi}{2} + \left(k_0 R - \frac{\pi}{2}\right) + \frac{k^2}{2k_0} R\right]$$

$$\simeq \left(k_0 + \frac{k^2}{2k_0}\right) (-) \tan\left[\frac{\beta}{k_0} + \frac{k^2 + \beta^2}{2k_0} R\right]$$

$$\simeq (-) k_0 \left(\beta/k_0 + \frac{k^2+\beta^2}{2k_0} R\right) \simeq -\beta - \frac{1}{2}(\beta^2 + k^2) R$$

$$\Rightarrow \quad \frac{kR\cot\delta - (kR)^2}{1 + kR\cot\delta} = -\beta R - \frac{1}{2}(\beta^2 + k^2)R^2$$

$$kR\cot\delta = \left[-\beta R - \frac{1}{2}(\beta^2+k^2)R^2\right](1 + kR\cot\delta] + (kR)^2$$

① zero-th $\quad kR\cot\delta = -\beta R$

② next order $\quad kR\cot\delta = (-\beta R - \frac{1}{2}(\beta^2+k^2)R^2)(1-\beta R) + (kR)^2$

$$\simeq -\beta R + \frac{1}{2}(k^2+\beta^2)R^2$$

$$\Rightarrow \quad \boxed{k\cot\delta = -\underbrace{\left(\beta - \frac{\beta^2}{2}R\right)}_{\frac{1}{a_0}} + \frac{k^2}{2}R}$$