lect 27 Lippman – Schwinger equation; Born approximation

- We will introduce another method of integration equation to treat the scattering amplitude, without ~~to~~ doing partial wave. In order to solve the Schringer equation

$$(\nabla^2 + k^2)\psi(r) = \frac{2m}{\hbar^2} V(r)\,\psi(r), \text{ under the boundary condition}$$

$$\psi(r) \xrightarrow{r\to\infty} e^{i\vec{k}\cdot\vec{r}} + f(\theta,\varphi)\,\frac{e^{ikr}}{r}.$$

Define Green function statisfying $(\nabla^2 + k^2)\,G(r,r') = \delta(r-r')$

$\Rightarrow \psi(r) = \frac{2m}{\hbar^2}\int dr'\, G(r,r')\,V(r')\,\psi(r')$ satisfies the Schrödinger eq.

- check $(\nabla^2+k^2)\,\psi(r) = \frac{2m}{\hbar^2}\int dr'\,(\nabla^2+k^2)\,G(r,r')\,V(r')\,\psi(r') = \frac{2m}{\hbar^2}\int dr'\,\delta(r-r')\,V(r')\,\psi(r')$

$$= \frac{2m}{\hbar^2} V(r)\,\psi(r).$$

we can also add the homogeous part $\psi^0(r)$ satisfying $(\nabla^2+k^2)\,\psi^0(r) = 0$ ↑ plane wave

$\Rightarrow$ Scattering problem reduces to the integral equation

$$\boxed{\psi(r) = e^{i\vec{k}\cdot\vec{r}} + \frac{2m}{\hbar^2}\int dr'\, G(r,r')\,V(r')\,\psi(r')}$$

↑ Lippman – Schwinger Eq.

- The next question is how to determine $G(r,r')$ to match the outgoing wave boundary condition $\psi_{sc}(r) = \frac{2m}{\hbar^2}\int dr'\, G(r,r')\,V(r')\,\psi(r') \xrightarrow{r\to+\infty} f(\theta,\varphi)\,\frac{e^{ikr}}{r}$.

due to the translational symmetry $G(r, r') = G(r-r')$

$$G(r-r') = \int d^3\vec{q}\; e^{i\vec{q}\cdot(\vec{r}-\vec{r}')}\, G(\vec{q})$$

$$\Rightarrow\ (-q^2 + k^2)\, G(\vec{q}) = \frac{1}{(2\pi)^3} \quad \text{or} \quad G(\vec{q}) = \frac{1}{(2\pi)^3}\,\frac{-1}{q^2-k^2}$$

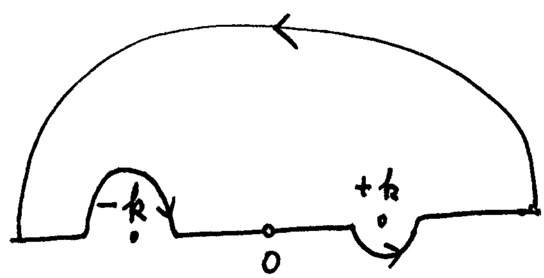
$$G(\vec{r}-\vec{r}') = -\frac{1}{(2\pi)^3}\int d^3\vec{q}\;\frac{1}{q^2-k^2}\, e^{i\vec{q}\cdot(\vec{r}-\vec{r}')}$$

$$= -\frac{1}{(2\pi)^3}\int_0^{\infty} q^2 dq \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi\; \frac{e^{iq|\vec{r}-\vec{r}'|\cos\theta}}{q^2-k^2} \qquad R = |\vec{r}-\vec{r}'|$$

$$= -\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} \frac{1}{iR}\, dq \cdot \frac{q\, e^{iqR}}{q^2-k^2}$$

, there's two first order poles $q = \pm k$

we need to decide the integral contour



$$= -\frac{2\pi i}{(2\pi)^2\, iR}\; \frac{q\, e^{iqR}}{q+k}\Bigg|_{q=k} = -\frac{1}{4\pi R}\, e^{ikR}$$

Another equivalent way is to add a small imaginary part to the Green function as $\frac{-1}{q^2-k^2-i\eta}\,\frac{1}{(2\pi)^3}$

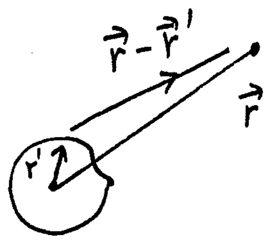
$$\Rightarrow\ \boxed{\psi(r) = e^{i\vec{k}_i\cdot\vec{r}} - \frac{m}{2\pi\hbar^2}\int d^3\vec{r}'\;\frac{e^{ik|r-r'|}}{|r-r'|}\, V(r')\,\psi(r')}$$

First order approx: (Born approximation)

$$\psi(r) = e^{i\vec{k}\cdot\vec{r}} - \frac{m}{2\pi\hbar^2}\int d^3\vec{r}\,' \, \frac{e^{ik|r-r'|}}{|r-r'|} \, V(r') \, e^{i\vec{k}\cdot\vec{r}\,'}$$

due to the short range of $V(r')$, $|r-r'|$ can be approximated as r in the denominator. But for the phase factor $e^{ik|r-r'|} \approx e^{ikr(1-\vec{r}\cdot\vec{r}\,'/r^2)}$

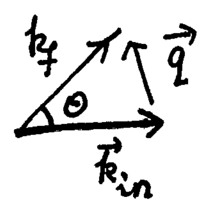
$$= e^{ikr}\, e^{-i\vec{k}_f\cdot\vec{r}\,'} \quad \text{where } \vec{k}_f = k\frac{\vec{r}}{r}$$



$$|\vec{r}-\vec{r}\,'| = (r^2 + r'^2 - 2\vec{r}\cdot\vec{r}\,')^{1/2} = r(1 - \vec{r}\cdot\vec{r}\,'/r^2)$$

$$\psi_{scattering}(\vec{r}) \xrightarrow{r\to\infty} -\frac{m e^{ikr}}{2\pi\hbar^2 r}\int d^3r' \, e^{-i(\vec{k}_f-\vec{k}_i)\cdot\vec{r}\,'} \, V(\vec{r}\,')$$

$$= -\frac{m e^{ikr}}{2\pi\hbar^2 r} V(\vec{k}_f - \vec{k}_i)$$



$\vec{q} = \vec{k}_f - \vec{k}$

$$\Rightarrow \psi(r) = e^{i\vec{k}_i\cdot\vec{r}} - \frac{e^{ikr}}{r}\frac{m}{2\pi\hbar^2}V(\vec{k}_f - \vec{k}) \Rightarrow f(\vec{k}_i, \vec{k}_f) = -\frac{m}{2\pi\hbar^2}V(\vec{k}_f - \vec{k}_i)$$

set $\vec{q} = \vec{k}_f - \vec{k}_i \Rightarrow$

$$V(q) = 2\pi\int r'^2 dr' \sin\theta d\theta \, V(r')\, e^{-iqr\cos\theta} = \frac{4\pi}{q}\int dr' \, r' \, V(r') \sin qr'$$

$$\Rightarrow f(\theta) = -\frac{2m}{\hbar^2 q}\int_0^\infty r' \, V(r') \sin qr' \, dr'$$

set k_{in} along the z-axis, $\Rightarrow q = 2k\sin\frac{\theta}{2}$

$$\Rightarrow \boxed{\sigma(\theta) = |f(\theta)|^2 = \frac{4\mu^2}{\hbar^4 q^2}\left|\int_0^{+\infty} r' V(r') \sin qr' \, dr'\right|^2,} \quad q = 2k\sin\frac{\theta}{2}$$

§ The condition for Born approximation:

- Born approx is essentially a perturbation theory, we need $|\psi_{scattering}| \ll e^{i\vec{k}\cdot\vec{r}}$ to justify it. At $r=0$, $V_{scattering}$ is ~~strongest~~, so the condition can be justified as $|\psi_{scattering}(0)| \ll 1$.

$$|\psi_{scattering}(0)| = \frac{m}{2\pi\hbar^2}\left|\int d^3r' \, \frac{e^{ikr'}}{r'} V(r') e^{i\vec{k}\cdot\vec{r}'}\right|$$

$$= \frac{2m}{\hbar^2 k}\left|\int_0^{+\infty} dr' \, e^{ikr'} V(r') \sin kr'\right| \ll 1$$

For low energy scattering, $\sin kr' \simeq kr'$ $\quad e^{ikr'} \simeq 1 \Rightarrow$

- $$\frac{2m}{\hbar^2}\left|\int_0^{+\infty} dr' \, r V(r')\right| \ll 1.$$ Let us conside $V(r')$ has range r_0 and strength V_0

$$\Rightarrow \frac{m}{\hbar^2}|V_0| r_0^2 \ll 1 \quad \text{i.e.} \quad \boxed{|V_0| \ll \frac{\hbar^2}{m r_0^2}}$$

For high energy scattering, $(kr_0 \gg 1)$

$$e^{ikr'} \sin kr' = \cos kr' \sin kr' + i \sin^2 kr'$$

the first term vanishes, and the second term has an averged value of $1/2$

$$\Rightarrow \quad \frac{m}{\hbar^2 k} r_0 |V_0| \ll 1$$

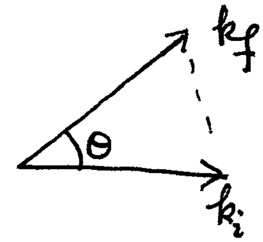
- or $$\boxed{|V_0| \lesssim \frac{\hbar^2 k}{m r_0} = \frac{\hbar^2 (kr_0)}{m r_0^2}}$$

Thus if Born approx is valid in the low energy secter, it's also valid in the high energy secter.

§ phase shift δ_ℓ from Born approximation

$$f(\theta) = -\frac{2m}{\hbar^2}\frac{1}{|\vec{k}_f-\vec{k}_i|}\int_0^\infty r' V(r') \sin|\vec{k}_f-\vec{k}_i| r' dr'$$

according to
$$\frac{\sin|\vec{k}_f-\vec{k}_i|r}{|\vec{k}_f-\vec{k}_i|r} = \sum_{\ell=0}^{\infty}(2\ell+1)\, j_\ell^2(kr)\, P_\ell(\cos\theta)$$



$$f(\theta) = -\frac{2m}{\hbar^2}\sum_{\ell=0}^{\infty}(2\ell+1) P_\ell(\cos\theta)\int_0^\infty r'^2 V(r')\, j_\ell^2(kr')\, dr'$$

$$= -\frac{2m}{\hbar^2}\sum_{\ell=0}^{\infty} Y_{\ell 0}(\theta)\sqrt{4\pi(2\ell+1)}\int_0^\infty r'^2 V(r')\, j_\ell^2(kr')\, dr'$$

c.f. Partial wave
$$f(\theta) = \sum_\ell \frac{e^{i\delta_\ell}}{k}\sin\delta_\ell\sqrt{4\pi(2\ell+1)}\; Y_{\ell 0}(\theta)$$

For small δ_ℓ, we have $e^{i\delta_\ell}\sim 1$, $\sin\delta_\ell \sim \delta_\ell$

$$\Rightarrow \quad -\frac{2m}{\hbar^2}\int_0^\infty r'^2 V(r')\, j_\ell^2(kr')\, dr' = \delta_\ell/k$$

$$\delta_\ell = -\frac{2mk}{\hbar^2}\int_0^\infty V(r')\, j_\ell^2(kr')\, r'^2 dr'$$

if $V(r)$ is short range, $j_\ell(kr) \xrightarrow{kr\to 0} \frac{(kr)^\ell}{(2\ell+1)!!}$

$$\Rightarrow \boxed{\delta_\ell \approx \frac{-2mkV_0}{\hbar^2}\int_0^{r_0}\frac{k^{2\ell} r^{2\ell}}{(2\ell+1)!!} r^2 dr \;\propto\; k^{2\ell+1}}$$

§ Coulomb scattering:

- Rigoursly speaking, due to long range nature of Culomb scattering, the boundary condition of $e^{ikr} + f(\theta,\varphi)\frac{e^{ikr}}{r}$ does not apply. And also $\int_0^{+\infty} r V(r) dr \to +\infty$, so the condition for Born approximation also fails.

we formally treat Coulomb potential as Yukawa potential

$$V(r) = \frac{\kappa}{r} e^{-\alpha r} \quad \text{as} \quad \alpha \to 0.$$ Born condition applies

if $\frac{2m}{\hbar^2}\left|\int_0^\infty r V(r) dr\right| = \frac{2m\kappa}{\hbar^2\alpha} << 1$.

- For Yukawa potential,

$$f(\theta) = -\frac{m}{2\pi\hbar^2} V(\vec{k}_f - \vec{k}_i) = -\frac{m}{2\pi\hbar^2}\frac{4\pi\kappa}{|k_f - k_i|^2 + \alpha^2} = -\frac{2m\kappa}{\hbar^2}\frac{1}{4k^2\sin^2\frac{\theta}{2} + \alpha^2}$$

$$\sigma(\theta) = |f(\theta)|^2 = \frac{4m^2\kappa^2}{\hbar^4}\frac{1}{(4k^2\sin^2\frac{\theta}{2} + \alpha^2)^2} \xrightarrow{\alpha\to 0} \frac{4m^2\kappa^2}{16\hbar^4 k^4 \sin^4\theta/2} = \frac{\kappa^2}{16E^2\sin^4\theta/2}$$

which is just the Rutherfold formula.

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