3.1 (a) $H = -J\sum_{\langle ij\rangle} \sigma_i\sigma_j - H\sum_{\langle ij\rangle} (\sigma_i+\sigma_j)/2$

$$Z = \mathrm{Tr}\left(e^{-\beta H}\right) = \mathrm{Tr}(T^N)$$

$$T = \begin{pmatrix} e^{\beta(J+H)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J+H)} \end{pmatrix} \qquad \begin{array}{l} k \equiv \beta J \\ h \equiv \beta H \end{array}$$

$$\therefore T = \begin{pmatrix} e^{k+h} & e^{-k} \\ e^{-k} & e^{k-h} \end{pmatrix}$$

Solve for eigen values

$$(e^{k+h}-\lambda)(e^{k-h}-\lambda) = e^{-2k}$$

$$\therefore \lambda_\pm = e^{k}\cosh(h) \pm \sqrt{e^{2k}\sinh^2(h)+e^{-2k}}$$

the corresponding eigenvectors are.

$$u^+ = \begin{pmatrix} e^{-k} \\ -e^{k}\sinh(h)+\sqrt{e^{2k}\sinh^2(h)+e^{-2k}} \end{pmatrix}$$

$$u^- = \begin{pmatrix} -e^{k}\sinh(h)+\sqrt{e^{2k}\sinh^2(h)+e^{-2k}} \\ -e^{-k} \end{pmatrix}$$

make $\cot(2\phi) = e^{2k}\sinh(h)$

$$-e^{k}\sinh+\sqrt{e^{2k}\sinh^2(h)+e^{-2k}}$$

$$= e^{-k}\left(-e^{2k}\sinh+\sqrt{e^{4k}\sinh^2(h)+1}\right) = e^{-k}\left(-\cot(2\phi)+\sqrt{\cot^2(2\phi)+1}\right)$$

$$= e^{-k}(-\cos 2\phi+1)/\sin 2\phi = e^{-k}\tan\phi$$

After normalization

$$\tilde{u}^+ = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \quad \tilde{u}^- = \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix}$$

$$S = \begin{pmatrix} \cos\phi & -\sin\phi \\ +\sin\phi & \cos\phi \end{pmatrix} \quad S^{-1} = \begin{pmatrix} \cos\phi & +\sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

$$S^{-1} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} S = \begin{pmatrix} \cos 2\phi & -\sin 2\phi \\ -\sin 2\phi & -\cos 2\phi \end{pmatrix}$$

$$\langle S_i \rangle = \mathrm{Tr}\left(T^i \sigma_{iz} T^{N-i} \right) / \mathrm{Tr}(T^N)$$

$$= \mathrm{Tr}\left((S^{-1}TS)^i (S^{-1}\sigma_{iz} S)(S^{-1}TS)^{N-i} \right) / \mathrm{Tr}(T^N)$$

$$= \mathrm{Tr}\left(\begin{pmatrix} \lambda^+ & \\ & \lambda^- \end{pmatrix}^N \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & \cos 2\phi \end{pmatrix} \right) / \mathrm{Tr}(T^N)$$

$N \to \infty$

$$= \cos 2\phi$$

$$\langle S_i S_{i+j} \rangle = \mathrm{Tr}\left(T^i \sigma_{iz} T^j \sigma_{i+j,z} T^{N-i-j} \right) / \mathrm{Tr}(T^N)$$

$$= \mathrm{Tr}\left(\begin{pmatrix} \lambda^+ & \\ & \lambda^- \end{pmatrix}^i \begin{pmatrix} \cos 2\phi & -\sin 2\phi \\ -\sin 2\phi & -\cos 2\phi \end{pmatrix} \begin{pmatrix} \lambda^+ & \\ & \lambda^- \end{pmatrix}^j \begin{pmatrix} \cos 2\phi & -\sin 2\phi \\ -\sin 2\phi & -\cos 2\phi \end{pmatrix} \begin{pmatrix} \lambda^+ & \\ & \lambda^- \end{pmatrix}^{N-i-j} \right) / \mathrm{Tr}(T^N)$$

$N \to \infty$

$$= \cos 2\phi \cos 2\phi + \sin^2 2\phi \left(\frac{\lambda^-}{\lambda^+} \right)^j$$

$$\therefore \langle S_i S_{i+j} \rangle - \langle S_i \rangle \langle S_{i+j} \rangle$$

$$= \cos 2\phi \cos 2\phi + \sin^2 2\phi \left(\frac{\lambda^-}{\lambda^+} \right)^j - \cos^2 2\phi = \sin^2 2\phi \left(\frac{\lambda^-}{\lambda^+} \right)^j$$

(c). $M = N \cos 2\phi \qquad (M = N\langle S \rangle)$

$$\chi = \partial M / \partial H = +\beta\, \partial M / \partial h = -\beta N\, 2 \sin 2\phi \frac{\partial \phi}{\partial h}$$

$$= -\beta N 2 \sin 2\phi\, \partial\left(\mathrm{arc\,cot}\, e^{2K} \sinh(h)\right) / \partial h \quad /2$$

$$= +\beta N \sin^3 2\phi\, e^{2K} \cosh(h)$$

$$\sum_j C_{ij} = \sin^2 2\phi \left(2 \frac{1}{1-\frac{\lambda^-}{\lambda^+}} - 1 \right)$$

$$= \sin^2 2\phi \left(\frac{\lambda^+ + \lambda^-}{\lambda^+ - \lambda^-} \right) = \sin^2 2\phi \; e^{2K} \cosh(h) \quad \checkmark$$

(d). there is no J term between 1. and N site, but H term is still there

∴ the Transfer matrix at the boundary need be modified

$$T_b = \begin{pmatrix} e^{h} & \phi \\ \phi & e^{-h} \end{pmatrix}$$

$$Z = \mathrm{Tr}\,(T^{N-1} T_b)$$

$$= \mathrm{Tr}\left(\begin{pmatrix} \lambda^{+(N-1)} & \\ & \lambda^{-(N-1)} \end{pmatrix} \begin{pmatrix} \cos\phi & +\sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} e^{h} & 1 \\ 1 & e^{-h} \end{pmatrix} \begin{pmatrix} \cos\phi & +\sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \right)$$

$$= \lambda^{+(N-1)} e^{-h} (-e^{h}\cos\phi + \sin\phi)^2 + \lambda^{-(N-1)} (\cos\phi + e^{h}\sin\phi)^2 e^{-h}$$

$$F = -k_B T \ln Z = -k_B T \left(N \log\lambda^+ + \ln\left(e^{-h}(+e^{h}\cos\phi + \sin\phi)^2/\lambda^+ + (e^{-h}(\cos\phi + e^{h}\sin\phi)^2/\lambda^- \cdot \left(\frac{\lambda^-}{\lambda^+}\right)^N \right) \right)$$

$$= -k_B T \left(N \log\lambda^+ + \ln\left(e^{-h}(+e^{h}\cos\phi + \sin\phi)^2/\lambda^+\right) + \ln\left(1 + \frac{(\cos\phi + e^{h}\sin\phi)^2 \lambda^+}{(+\cos\phi\, e^{h} + \sin\phi)^2 \lambda^-} \left(\frac{\lambda^-}{\lambda^+}\right)^N \right) \right)$$

$$\therefore \quad f_b(h,K) = \lambda^+ \cdot (-k_B T)$$

$$f_s(h,K) = \ln\left(e^{-h}(e^{+h}\cos\phi + \sin\phi)^2/\lambda^+ \right) (-k_B T)$$

$$F_{fs} = \ln\left(1 + \frac{(\cos\phi + e^{h}\sin\phi)^2 \lambda^+}{(+\cos\phi\, e^{h} + \sin\phi)^2 \lambda^-} \left(\frac{\lambda^-}{\lambda^+}\right)^N \right) (-k_B T).$$

$$\xrightarrow{N\to\infty} \; \sim \left(\frac{\lambda^-}{\lambda^+}\right)^N$$

$N \to 0, \ h \to 0$

$\cot(2\phi) = e^{2k}\sinh(h) = 0$

$\therefore \ \phi = +\frac{\pi}{4} \qquad \lambda^{+} = 2\cosh(K)$

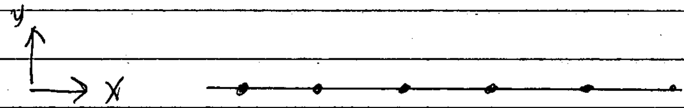
$\therefore \ F = -k_BT \, N \log \lambda^{+} + k_BT \log \lambda^{+} - k_BT \log 2$

$= -k_BT \, (N-1) \log 2\cosh K - k_BT \log 2$

This is indeed the free energy for open boundary chain

3.2 M row Ising chain

(1) $M = 1$.



In x direction, each spin couples to its neighbours

In y direction, each spin couples to itself (PBC).

$$Z = \sum_{\{S_i\}} e^{-\beta J \sum (S_i S_i + S_i S_{i+1})}$$

$$= \sum_{\{S_i\}} e^{-\beta J \sum_{\langle i,j\rangle} (\frac{1}{2}(S_i S_i + S_j S_j) + S_i S_j)}.$$

$\langle i,j\rangle$ is nearest neighbour.

We can define transfer matrix T_{ij}

$$T_{ij} = e^{-\beta J (\frac{1}{2}(S_i S_i + S_j S_j) + S_i S_j)}.$$

$Z = \mathrm{Tr}\, T^N$ $\quad$ T is 2 by 2 matrix

In matrix form, we have
$$T = \begin{array}{c c} & \begin{array}{cc} \uparrow & \downarrow \end{array} \\ \begin{array}{c} \uparrow \\ \downarrow \end{array} & \begin{pmatrix} e^{2K} & 1 \\ 1 & e^{2K} \end{pmatrix} \end{array}$$

$K = -\beta J$.

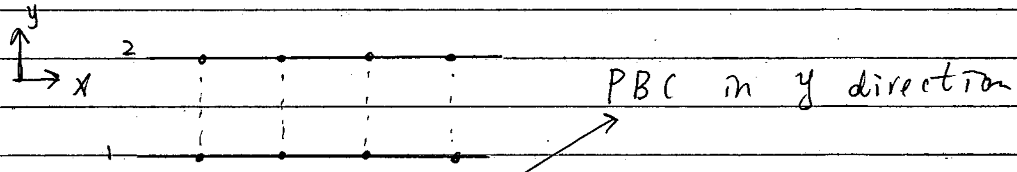
define $x = e^K$ $\quad$ $T = \begin{pmatrix} x^2 & 1 \\ 1 & x^2 \end{pmatrix}$

solving for eigenvalues:

$(x^2 - \lambda)(x^2 - \lambda) = 1$.

$\therefore \lambda = x^2 \pm 1$.

(2) $M = 2$

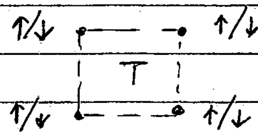


$$Z = \sum e^{-\beta J \sum_i 2 S_{i,1} S_{i,2} + \sum_{\langle ij \rangle} S_{i,1} S_{j,1} + S_{i,2} S_{j,2}}$$

$$= \sum e^{-\beta J \sum_{\langle ij \rangle} (S_{i,1} S_{j,2} + S_{j,1} S_{j,2}) + S_{i,1} S_{j,1} + S_{i,2} S_{j,2}}$$

$$= \mathrm{Tr}(T^N)$$

T is defined in the square.



T can be written as a matrix T_{ij}, where i describes the two spin on the left and j describes the two spins on the right. Therefore T is a 4x4 matrix

1. $\begin{array}{c}\uparrow\\\uparrow\end{array}$ 2. $\begin{array}{c}\uparrow\\\downarrow\end{array}$ 3. $\begin{array}{c}\downarrow\\\uparrow\end{array}$ 4. $\begin{array}{c}\downarrow\\\downarrow\end{array}$

$T_{11} \rightarrow \begin{array}{cc}\uparrow & \uparrow\\ \uparrow & \uparrow\end{array} = e^{K(1+1+1+1)} = e^{4K} = x^4$

$T_{22} \rightarrow \begin{array}{cc}\uparrow & \uparrow\\ \downarrow & \downarrow\end{array} = e^{K(-1-1+1+1)} = 1$

$T_{33} \rightarrow \begin{array}{cc}\downarrow & \downarrow\\ \uparrow & \uparrow\end{array} = 1$

$T_{44} \rightarrow \begin{array}{cc}\downarrow & \downarrow\\ \downarrow & \downarrow\end{array} = e^{4K} = x^4$

$T_{12} \rightarrow \begin{array}{cc}\uparrow & \uparrow\\ \uparrow & \downarrow\end{array} = e^{K(1-1+1-1)} = 1$ $\qquad T_{21} = T_{12}$

$T_{13} \rightarrow \begin{array}{cc}\uparrow & \downarrow\\ \downarrow & \uparrow\end{array} = e^{K(1-1-1+1)} = 1$ $\qquad T_{31} = T_{13}$

$T_{14} \rightarrow \begin{array}{cc}\uparrow & \downarrow\\ \uparrow & \downarrow\end{array} = e^{K(1+1-1-1)} = 1$ $\qquad T_{14} = T_{41}$

$$T_{23} \rightarrow \begin{pmatrix} \uparrow & \downarrow \\ \downarrow & \uparrow \end{pmatrix} = e^{K(-1-1-1-1)} = X^{-4} \qquad T_{32} = T_{23}$$

$$T_{24} \rightarrow \begin{pmatrix} \uparrow & \downarrow \\ \downarrow & \downarrow \end{pmatrix} = e^{K(-1+1-1+1)} = 1 \qquad T_{42} = T_{24}$$

$$T_{34} \rightarrow \begin{pmatrix} \downarrow & \downarrow \\ \uparrow & \downarrow \end{pmatrix} = e^{K(-1+1+1-1)} = 1 \qquad T_{34} = T_{43}.$$

$$\therefore T = \begin{pmatrix} X^4 & 1 & 1 & 1 \\ 1 & 1 & X^{-4} & 1 \\ 1 & X^{-4} & 1 & 1 \\ 1 & 1 & 1 & X^4 \end{pmatrix}$$

We need solve for eigenvalues

first apply an unitary transformation

$$T' = UTU^+ = \begin{pmatrix} X^4 & 1 & 1 & 1 \\ 1 & X^4 & 1 & 1 \\ 1 & 1 & 1 & X^{-4} \\ 1 & 1 & X^{-4} & 1 \end{pmatrix}.$$

$$= X^4 \sigma_0 \otimes \left(\frac{\tau_0 + \tau_3}{2}\right) + X^{-4} \sigma_1 \otimes \frac{(\tau_0 - \tau_3)}{2}$$

$$+ \sigma_1 \otimes \frac{(\tau_0+\tau_3)}{2} + \sigma_0 \otimes \left(\frac{\tau_0-\tau_3}{2}\right) + (\sigma_0 + \sigma_1) \otimes \tau_1$$

we are free to rotate σ_1 to σ_3.

$$T'' = X^4 \sigma_0 \otimes \frac{(\tau_0+\tau_3)}{2} + X^{-4} \sigma_3 \otimes \left(\frac{\tau_0 - \tau_3}{2}\right) + \sigma_3 \otimes \left(\frac{\tau_0+\tau_3}{2}\right)$$

$$\sigma_0 \otimes \frac{(\tau_0-\tau_3)}{2} + (\sigma_0 + \sigma_3) \otimes \tau_1.$$

T'' is diagonal in σ

$$\therefore T''^{+} = (X^4 + X^{-4}) \frac{\tau_0}{2} + (X^4 - X^{-4}) \frac{\tau_3}{2} + \tau_0 + 2\tau_1 \qquad (\sigma_3 = 1)$$

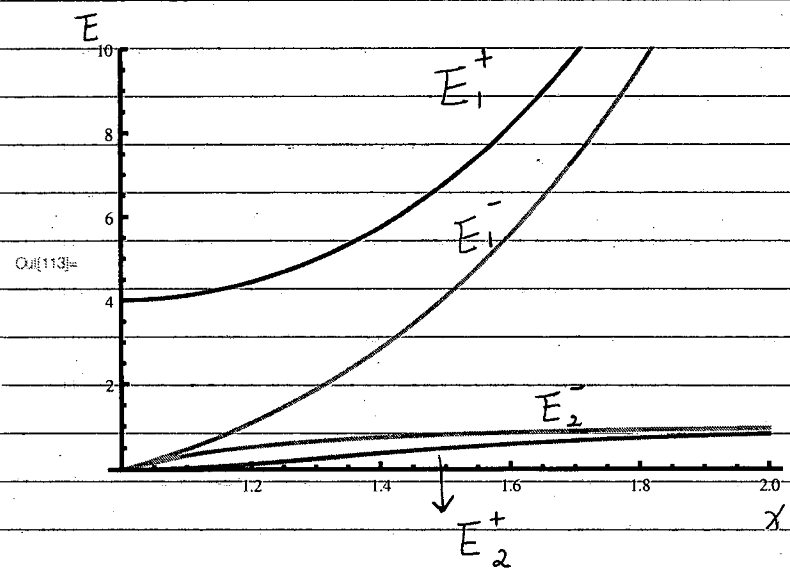
$$T''^{-} = (X^4 - X^{-4}) \frac{\tau_0}{2} + (X^4 + X^{-4}) \frac{\tau_3}{2} - \frac{\tau_0+\tau_3}{2} + \frac{\tau_0 - \tau_3}{2} \quad (\sigma_3 = -1)$$

using pauli matrix trick we can obtain the eigenvalues for T''^{+}

$$E^+_{1,2} = \frac{1}{2}\left[(X^4 + 2 + X^{-4}) \pm \sqrt{X^8 + X^{-8} + 14}\right]$$

T''^{-} is already diagnol with $\tau_3 = \pm 1$

$$\therefore E^-_1 = X^4 - 1 \quad , \quad E^-_2 = -X^{-4} + 1.$$



Out[113]=
E
10
8
6
4
2
1.2
1.4
1.6
1.8
2.0
x

3.3. (a) $y_i = x_i - (A^{-1})_{ij} B_j$

$\therefore x_i A_{ij} x_j = y_i A_{ij} y_j + 2 B_i y_i + B_i A^{-1}_{ij} B_j$

$x_i B_i = y_i B_i + B_i A_{ij}^{-1} B_j$

$\therefore -\frac{1}{2} x_i A_{ij} x_j + B_i x_i = \frac{1}{2} B_i (A^{-1})_{ij} B_j - \frac{1}{2} y_i A_{ij} y_j$.

$$\int \prod \frac{dx_i}{\sqrt{2\pi}} \exp(-\tfrac{1}{2} y_i A_{ij} y_j) \exp(\tfrac{1}{2} B_i (A^{-1})_{ij} B_j)$$

$$= \int \prod \frac{dy_i}{\sqrt{2\pi}} \exp(-\tfrac{1}{2} y_i A_{ij} y_j) \exp(\tfrac{1}{2} B_i (A^{-1})_{ij} B_j)$$

we can diagonalize A with unitary transform U.

$y_i A_{ij} y_j = \sum_i \tilde{y}_i D_i \tilde{y}_i$ $\quad D_i$ is eigenvalue.

$\tilde{y}_i = U y_i$ $\quad$ so the above equation becomes

$$\int \prod \frac{d\tilde{y}_i}{\sqrt{2\pi}} \exp(-\tfrac{1}{2} \sum \tilde{y}_i^2 D_i) \exp(\tfrac{1}{2} B_i (A^{-1})_{ij} B_j)$$

$$= \prod \int \frac{d\tilde{y}_i}{\sqrt{2\pi}} \exp(-\tilde{y}_i^2 D_i) \exp(\tfrac{1}{2} B_i (A^{-1})_{ij} B_j)$$

$$= \frac{1}{\sqrt{\prod D_i}} \exp(\tfrac{1}{2} B_i (A^{-1})_{ij} B_j) = \frac{1}{\sqrt{\det(A)}} \exp(\tfrac{1}{2} B_i (A^{-1})_{ij} B_j)$$

(b) $Z = \sum_{\{S_i\}} e^{-\beta(\frac{1}{2}\sum_{ij} J_{ij} S_i S_j + \sum_i H_i S_i)}$

to use the identity above, we want to make $A^{-1} = \frac{1}{\beta} J$ and $B_i = \beta S_i$

However, we need make sure that J is invertible.

This can be achieved by adding constant diagonal term.

$$Z = \sum_{\{S_i\}} e^{-\beta(\frac{1}{2} \tilde{J}_{ij} S_i S_j - \frac{1}{2} \sum \tilde{J}_{ii} S_i S_j + \sum_i H_i S_i)}$$

$$= \sum_{\{S_i\}} e^{-\beta(\frac{1}{2} S_i \tilde{J}_{ij} S_j - \frac{1}{2} \tilde{J}_{ii} + \sum_i H_i S_i}$$

$$\sim \sum_{\{S_i\}} e^{-\beta(\frac{1}{2} S_i \tilde{J}_{ij} S_j + H_i S_i)}$$ now $\tilde{J}_{ij}$ is invertible.

(c) $A^{-1} = \frac{1}{\beta} J \quad B_i = \beta S_i$

$$e^{\beta \frac{1}{2} S_i J_{ij} S_j + \beta H_i S_i}$$

$$\sim \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\varphi_i \, e^{-\frac{\beta}{2} \varphi_i J^{-1}_{ij} \varphi_j + \beta(\varphi_i + H_i) S_i}$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\varphi_i \, e^{-\frac{\beta}{2}(\varphi_i - H_i) J^{-1}_{ij} (\varphi_j - H_j) + \beta \varphi_i S_i}$$

$$Z = \sum_{\{S = \pm 1\}} \int_{-\infty}^{\infty} \prod d\varphi_i \, e^{-\frac{\beta}{2}(\varphi_i - H_i) J^{-1}_{ij} (\varphi_j - H_j) + \beta \varphi_i S_i}$$

$$= \int_{-\infty}^{\infty} \prod d\varphi_i \, e^{-\frac{\beta}{2}(\varphi_i - H_i) J^{-1}_{ij} (\varphi_j - H_j)} 2\cosh(\beta \varphi_i).$$

$$= \int_{-\infty}^{\infty} \prod d\varphi_i \, e^{\beta S\{\varphi_i\}}$$

$$S\{\varphi_i\} = \frac{1}{2}(\varphi_i - H_i) J^{-1}_{ij} (\varphi_j - H_j) - \frac{1}{\beta} \sum_i \log(2\cosh(\beta \varphi_i)).$$

(d). now we need find $\{\varphi_i\}$ minimize S:

$$\therefore \frac{\partial S}{\partial \varphi_i} = 0$$

$$J^{-1}_{ij}(\varphi_j - H_j) - \cancel{\frac{1}{\beta}} \frac{\sinh(\beta \varphi_i)}{\cosh(\beta \varphi_i)} = 0.$$

$$\therefore J^{-1}_{ij}(\bar{\varphi}_j - H_j) - \cancel{\beta} \tanh(\beta \bar{\varphi}_i) = 0.$$

$$m_i = -\frac{\partial S}{\partial H_i} = J^{-1}_{ij}(\varphi_j - H_j) = \tanh(\beta \varphi_i)$$

$$\therefore \boxed{m_i = \tanh \beta \bar{\varphi}_i}$$

$$\because m_i = J^{-1}_{ij}(\bar{\varphi}_j - H_j).$$

$$\therefore H_i = -J_{ij} m_j + \varphi_i = -J_{ij} m_j + \frac{1}{\beta} \operatorname{arctanh}(m_i).$$

(e) $\mathcal{S} = \frac{1}{2}(\varphi_i - H_i) J^{-1}_{ij} (\varphi_j - H_j) - \frac{1}{\beta} \sum_i \log(2\cosh\beta\varphi_i)$

plug in $\varphi_i - H_i = J_{ik} m_k$.

$$\mathcal{S} = \frac{1}{2} J_{ik} m_k J^{-1}_{ij} J_{jl} m_l - \frac{1}{\beta}\sum_i \log(2\cosh\beta\varphi_i)$$

$$= \frac{1}{2} m_i J_{ij} m_j - \frac{1}{\beta}\sum \log(2\cosh\beta\varphi_i)$$

$\because \tanh(\beta\varphi_i) = m$

$$\therefore \frac{1}{\sqrt{1-m^2}} = \frac{1}{\sqrt{1-\tanh^2(\beta\varphi_i)}} = \cosh(\beta\varphi_i)$$

$$\therefore \mathcal{S} = \frac{1}{2} m_i J_{ij} m_j + \frac{1}{\beta}\sum_i \log(\sqrt{1-m^2}/2)$$

$$T = \mathcal{S} + \sum H_i m_i = -\frac{1}{2} m_i J_{ij} m_j + \frac{1}{\beta}\sum_i \log(\sqrt{1-m^2}/2) + \frac{1}{\beta}\sum_i \operatorname{arctanh}(m_i)\, m_i$$

$$\frac{\partial T}{\partial m_i} = -\frac{1}{2} J_{ij} m_j + \frac{1}{\beta}\operatorname{arctanh}(m_i) + \frac{1}{2\beta}\frac{-2m}{1-m^2} + \frac{1}{\beta}\frac{1}{1-m^2}$$

$$= -\frac{1}{2} J_{ij} m_j + \frac{1}{\beta}\operatorname{arctanh}(m_i) = H_i \quad \checkmark$$