$4-\epsilon$ expansion: Momentum space RG (I) – Gaussian

We begin with the action of the φ^4 theory (Ising class)

$$\mathcal{Z} = \int DS\; e^{-F(S)} \quad \text{with} \quad F(S) = \int d^d r \left\{ \left(\frac{1}{2}\nabla S\right)^2 + \frac{r_0}{2} S^2 + \frac{1}{4} u_0 S^4 - h_0 S \right\}.$$

where the coefficient γ_0 has been absorbed into the definition of the field S.

1) At $r^* = h^* = u^* = 0$, this is a free theory with power-law correlation.
We assume that this is a fixed point — called Gassian fixed point.
We will see that the phase transition at $d>4$ is controlled by this fixed point.

2) Nevertheless, we will see that at $d<4$, or define $\epsilon = 4-d > 0$.
We will find the Gassian fixed point is unstable to a new fixed point determined by the following RG equation

$$\frac{du}{d\ln b} = \epsilon u - A u^2, \qquad A \text{ is some } \text{~~constant~~ to be calculated}$$

$$\Rightarrow u_0^* = \frac{\epsilon}{A},$$ which is a nontrivial (non-Gaussian) fixed point!

Before we study non-Gaussian fixed point, let us consider the Gaussian fixed point.

§ Gaussian fixed point

$$F = \int d^d r \ \frac{1}{2}(\nabla S)^2 + \frac{1}{2} r_0 S^2$$ where $a = 1/\Lambda$ is the short distance cut off, and Λ is momentum cutoff.

define $S(r) = \int \frac{d^d k}{(2\pi)^d} e^{ikr} S(k)$, and $S(k) = \int d^d k \ e^{-ikr} S(r)$

$$\Rightarrow F = \int_{0<k<\Lambda} \frac{d^d k}{(2\pi)^d} \left[\frac{k^2}{2} + r_0\right] |S(k)|^2$$

① Divid fast and slow fields

fast: $S_>(k)$ with $\Lambda/l < k < \Lambda$; slow $S_<(k)$ with $k < \Lambda/l$

$l > 1$

② Integrate out fast modes, and only keep the slow modes. This ~~step~~ is easy for Gaussian model. because $S_>$ and $S_<$ decouple.

$$Z = \int DS_> \int DS_< \ e^{-\int_{0<k<\Lambda/l} \frac{d^d k}{(2\pi)^d} \left(\frac{k^2}{2} + r_0\right)|S(k)|^2} \cdot e^{-\int_{\Lambda/l<k<\Lambda} \frac{d^d k}{(2\pi)^d} \left(\frac{k^2}{2} + r_0\right)|S(k)|^2}$$

The integration over $S_>$ give rise to a constant independent of $S_<$

$$\Rightarrow \boxed{Z = \text{const} \cdot \int DS_< \ e^{-\int_{0<k<\Lambda/l} \frac{d^d k}{(2\pi)^d} \left(\frac{k^2}{2} + r_0\right) |S_<(k)|^2}}$$

③ However, we cannot directly compare the new model with Λ/l with the old model with the cut off Λ. We need to make the cut off to be the same.

We introduce $k' \equiv l k$ $\Rightarrow$

$$Z = \int D S_< \; e^{-\int_{0<k'<\Lambda} \frac{d^d k'}{(2\pi)^d} l^{-d} \left(\frac{k'^2}{2} l^{-2} + r_0\right) |S_<(k'/l)|^2}$$

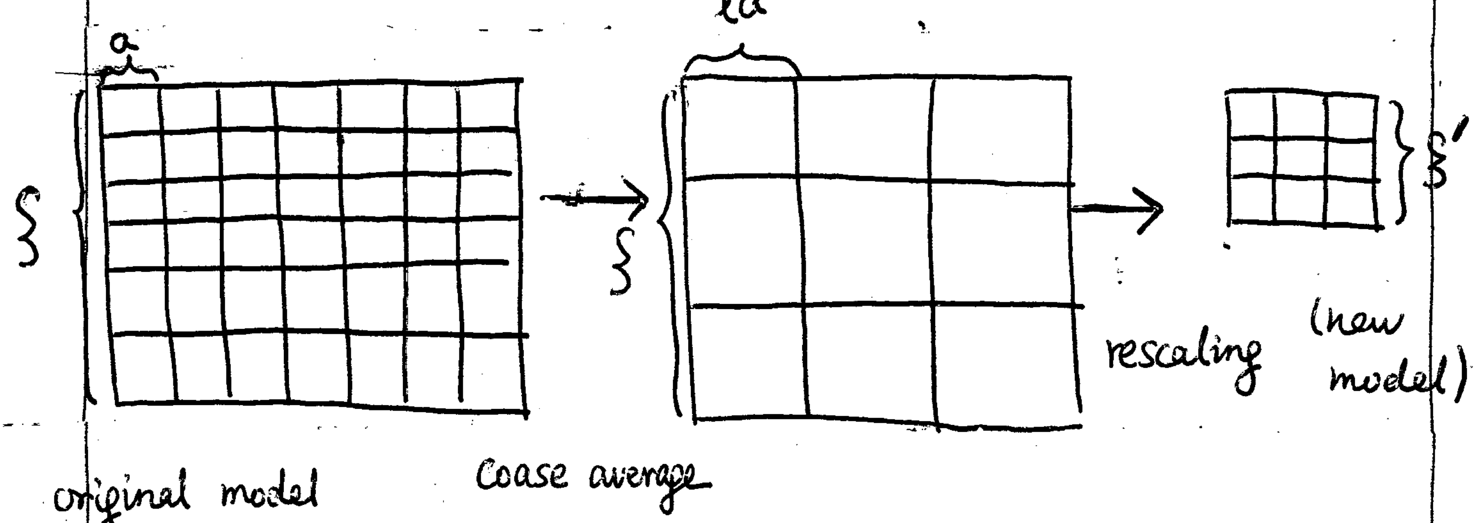
Now there's a new trouble that the coefficient of k'^2 is no longer 1.

Thus we define $S'(k') = l^{-\frac{d}{2}-1} S_<(k'/l)$ $\Rightarrow$ (wavefunction renormalization)

$$\boxed{Z = \int D S' \; e^{-\int_{0<k'<\Lambda} \frac{d^d k'}{(2\pi)^d} (k'^2 + r_0 l^2) |S'(k')|^2}}$$

Now, we can compare these two models $\boxed{r_l = r_0 l^2}$.

What are we doing here? We enlarged the miscrosopic unit from $a \to la$. The same correlation length ξ, when measured by the unit la, it values is smaller, i.e $\frac{\xi}{la} = \frac{1}{l}\left(\frac{\xi}{a}\right)$.



Thus the new model has shorter correlation length by a factor of the rescaling factor l, and thus is even further pushed away from critical point!

$$r(l) = l^2 r_0 \Rightarrow \frac{dr(l)}{dl} = \frac{2}{l} r(l)$$

$$\Rightarrow \frac{dr(l)}{dl/l} = 2r(l) \quad \text{or} \quad \boxed{\frac{dr}{d\ln l} = 2r}.$$

So the eigenvalue $y_r = 2$.

Let's show that this eigenvalue correspond to ν

The RG should stop at where $\boxed{r_0 l^{y_r} \sim \Lambda \sim 1/a}$ otherwise it doesn't make more sense. This basically the same as seting $\boxed{l = \frac{\xi_0(t)}{a}}$

thus $r_0 \left(\frac{\xi_0(t)}{a}\right)^{y_r} \sim 1/a$ and $r_0 \sim t$

$$\Rightarrow \frac{\xi_0(t)}{a} \sim t^{-1/y_r} \Rightarrow \boxed{\nu = 1/y_r}$$

at Gaussian level. $\nu = 1/2$.

2) how about external field?

$\Delta F = h \int d^d r\, S(r)$. This term correspond to the Fourier componet of $k=0$, and is not affected by integrating out the fast modes.

$$\Delta F = h\, S(q=0),$$

this term is affected by the wavefunction renormalization

$$\Rightarrow h\, l^{\frac{d}{2}+1} S'(q'=0) \Rightarrow h_l = l^{1+d/2} h$$

$$\Rightarrow \frac{dh}{dl} = (1+d/2)\frac{h}{l} \quad \text{and} \quad \boxed{\frac{dh}{d\ln l} = (1+d/2)\, h}$$

$\Rightarrow$ the eigenvalue of h: $\quad y_h = 1 + d/2$.

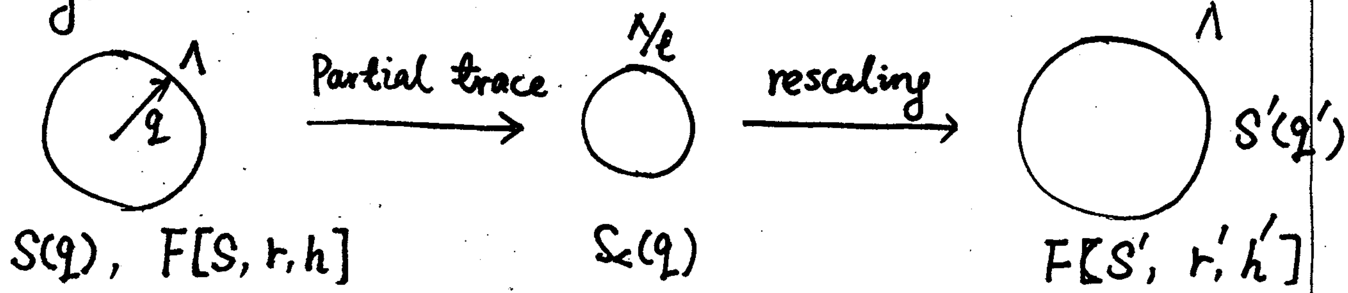
recall previous scaling analysis, we have

$$G(R, t, h) = R^{-2(d-y_h)} \, F(R t^{\frac{1}{y_t}}, h t^{-y_h/y_t})$$

$$= R^{-(d-2+\eta)} \, F(R t^{\nu}, h t^{-\Delta})$$

$\Rightarrow \boxed{\eta = 2y_h - (d+2)}$ $\qquad$ for Gaussian fixed point $\eta = 0$.

Summary:



At Gaussian level, the RG equation is already linear. For the genreral case, we need to linearize the RG equation around the fixed points, and find eigenvalues.

$$F(r, h, V) = F(r', h', V') + \text{smooth part}$$

$$\Rightarrow f(r,h) = \frac{F(r,h,V)}{V} = \frac{F(r',h',V')}{\ell^d V'} = \frac{1}{\ell^d} f'(r', h')$$

$$f(r,h) = 1/\ell^d \, f(\ell^{y_r} r, \ell^{y_h} h)$$

c.f. Scaling hypothesis $\quad f(t, h) = 1/\lambda \, f(\lambda^p t, \lambda^q h)$

$$\Rightarrow \begin{cases} \lambda = \ell^d \\ \lambda^p = \ell^{y_r} \\ \lambda^q = \ell^{y_h} \end{cases} \Rightarrow \begin{cases} p = y_r/d \\ q = y_h/d \end{cases}$$

cf. Scaling analysis, we have $\beta = \frac{1-q}{p}$, $\gamma = \frac{2q-1}{p}$, $\delta = \frac{q}{1-q}$, $\alpha = \frac{2p-1}{p}$.

we arrive at the following critical exponent for Gaussian model

$$\boxed{p = 2/d, \quad q = \frac{2+d}{2d}}$$

Gaussian:

	MF
$\alpha = 2 - d/2$	0
$\beta = \frac{d-2}{4}$	$1/2$
$\gamma = 1$	1
$\delta = \frac{d+2}{d-2}$	3
$\eta = 0$	0
$\nu = 1/2$	$1/2$.

Gaussian model's critical exponents = Mean field ones at $d=4$.

⊛ Dangerous irrelevant operator: interaction term.

As we can see at $d>4$, β and δ from naive Gaussian theory does not agree with MF from Landau's theory. The reason is that Gaussian model can only be applied at disordered phase, But β and δ are related to ordered phase or right at the critical point, at which the "u"-term although irrelevant but it cannot be neglected.

Let us consider the scaling form of the free energy

$$f_s(t, h, u_0) = \ell^{-d} f_s(t\ell^{y_t}, \ h\ell^{y_h}, u_0\ell^{y_u})$$

The Gaussian fixed point: $y_t = 2$, $y_h = 1 + d/2$, $\underline{y_u = 4-d}$ will explain later.

at $d > 4$, u_0 is irrelevant because $y_u < 0$.

$$M(t\ h, u_0) \equiv -\frac{1}{k_B T}\frac{\partial f}{\partial h} = \ell^{-d+y_h} \quad M(t y^{y_t}, h\ell^{y_h}, u_0\ell^{y_u})$$

Setting $h=0$, , and set $\ell = |t|^{-1/y_t}$

$$\boxed{M(t, 0, u_0) = |t|^{-(y_h-d)/y_t} \ M(-1, \ 0, \ u_0|t|^{-y_u/y_t})}$$

① if we simply set $u_0 = 0$, we would arrive

$$M(t, 0, u_0) \sim |t|^{-\beta} M(-1, 0, 0) \quad \text{with } \beta = \frac{y_h - d}{y_t} = \frac{d-2}{4}$$

which is wrong!

This is because from MF theory, we know $M(t, 0, u_0) \propto \sqrt{\frac{|t|}{u_0}} \propto u_0^{-1/2}$, and thus we cannot set $u_0 = 0$ in the ordered phase!

we define the function $M(-1, 0, u_0|t|^{-y_u/y_t}) \triangleq F_M\left(\frac{u_0}{|t|^{y_u/y_t}}\right)$,

but $F(x)$ is singular at $x = 0$, as $F(x) \propto x^{1/2}$, and this power of $1/2$ is obtained in the Landau theory. Then we have

$$M(t, 0, u_0) \sim |t|^{-(y_h-d)/y_t + y_u/2y_t} \sim |t|^{\beta}$$

$$\Rightarrow \beta = -\tfrac{1}{2}(1 - \tfrac{d}{2}) + \frac{4-d}{4} = 1/2. \quad \text{(which is correct!)}$$

$$\ell = h^{-1/y_h}$$

how about δ, set $t=0$, we have

$$M(0, h, u_0) = h^{-(y_h - d)/y_h} \, M(0, 1, u_0 h^{-y_u/y_h})$$

Again since from mean field, we know $M(0, h, u_0) \sim (h/u_0)^{+1/3}$,

we cannot set u_0 in the form of $M(0, 1, u_0 h^{-y_u/y_h})$ but rather

$$\Rightarrow M(0, h, u_0) \sim h^{-(y_h - d)/y_h} \cdot h^{\frac{1}{3} y_u/y_h}$$

$$\Rightarrow 1/\delta = -\frac{y_h - d}{y_h} + \frac{1}{3}\frac{y_u}{y_h} \Rightarrow \delta = 3.$$

Although we need to know $\beta = 1/2$ and $\delta = 3$, before doing scaling, after all we checked that the theory is self-consistent.