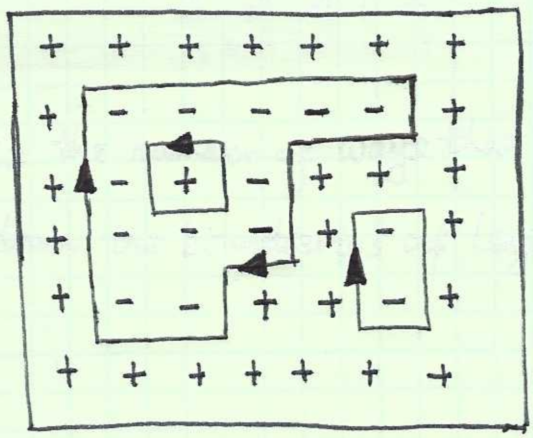
Existence of the PM-FM phase transition of the 2D Ising model

The kink argument in 1D is no-longer valid in 2D, because the domain wall becomes a line whose size can reach the order of L. (L is the length of the sample and $N = L^2$.) Naively, flip one spin cost the energy $2zJ$, where z is the coordination number. When $T \simeq zJ$, every site has the probability at the order of 1 to be flipped, $\Rightarrow T_c$ is at order of zJ. In fact, this is the mean-field T_c given by the Bragg-Williams approximation.

We follow Peierls and Griffiths to show the existance of low T ordered phase. In 2D, a general configuration can be represented by domain wall lines. We assume all spins on the surface are pinned to be spin up as an external field. The effects of the surface are expected to vanish as $L \to +\infty$.

Configurations $\xrightarrow{\text{map}}$ Closed loops.

domain Loop length b



The number of sites enclosed by this loop $\leq \left(\frac{b}{4}\right)^2$.

Denote $m(b)$ as the number of domain wall loops with length b

$m(b) \leq 3^{b-1} N$. $\longrightarrow$ The first bond can pick any position, as the next one has 3-direction to pick.

Thus the number of spin down

$$\langle N_{\downarrow} \rangle \leq \sum_{b} \left(\frac{b}{4}\right)^2 \cdot N\, 3^{b-1} \cdot \boxed{\text{Probility (loop of length } b)}.$$

All configuration = (configuration with the loop b) + configs without

Thus its probility $= \dfrac{e^{-2\beta b J}}{1 + e^{-2\beta b J}}$ where $2bJ$ is the energy cost

$$< e^{-2\beta b J}$$

$$\Rightarrow \frac{N_{\downarrow}}{N} \leq \sum_{b = 4, 6, \cdots} \left(\frac{b}{4}\right)^2 3^{b-1} e^{-2\beta \epsilon b} = \frac{x^2}{3(1-x)^3}\left(1 - \tfrac{3}{4}x + \frac{x^2}{4}\right)$$

with $x = 9 e^{-2\beta\epsilon}$

as β sufficiently large, or $x \to 0$, certainly we can have $\frac{N_{\downarrow}}{N} < \frac{1}{2}$, and thus we have ordered phase!

Infact, we can solve exactly T_c even without solve the partition function. — by Kramers-Wannier duality!

Kramer's and Wannier's duality

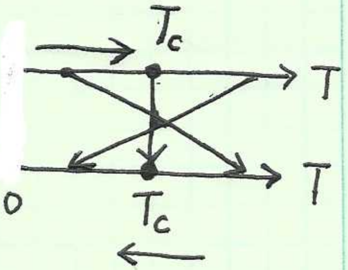
In the absenteof external field, the partition function of 2D Ising model of $N = n \times n$ sites can be mapped ~~t~~ to that of the same model but with a different ~~temperature~~: i.e

$$\frac{\ln Z(N,\beta)}{N} = \frac{\ln Z(N,\beta^*)}{N} - \sinh 2\beta^*$$

$\boxed{\beta = J/k_BT \text{ combined into one parameter.}}$

where $\beta^* = -\frac{1}{2}\ln\tanh\beta$, or $\sinh 2\beta \sinh 2\beta^* = 1$. Thus the low ~~temperature~~ phase is mapped to the high temperature phase. If we assume that there are only two phases and one transition, we argue that the critical ~~temperture~~ β_c is the self-dual point

i.e $\quad (\sinh 2\beta)_c^2 = 1$, or $\beta_c = \frac{1}{2}\ln(1+\sqrt{2})$



$$k_BT_c/J = \frac{2}{\ln(\sqrt{2}+1)} = 2.269.$$

In other word, 2D Ising model if self-dual, and we can solve T_c exactly even without solving the model! Next we will prove its self-duality.

$$Z(N,\beta) = \sum_{\{\sigma_i\}} e^{\beta \sum_{\langle ij\rangle} \sigma_i\sigma_j} = \sum_{\{\sigma_i\}} \prod_{\langle ij\rangle} e^{\beta\sigma_i\sigma_j}$$

$$e^{\beta\sigma_i\sigma_j} = \cosh\beta + \sigma_i\sigma_j \sinh\beta$$

with $C_0 = \cosh\beta$, $C_1 = \sinh\beta$

$$\Rightarrow Z(N,\beta) = \sum_{\{\sigma_i\}} \prod_{\langle ij\rangle} \sum_{k_{ij}=0}^{1} C_{k_{ij}} (\sigma_i\sigma_j)^{k_{ij}}$$

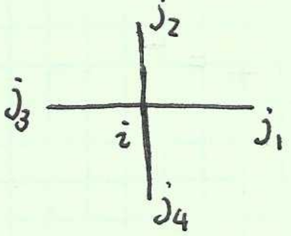
each $\langle ij \rangle$ is assigned an integer $k_{ij} = 0$ or 1 $\quad \langle ij \rangle$ is a bond

$$Z[N,\beta] = \sum_{\{\sigma_i\}} \sum_{\{k_{\langle ij \rangle}\}} \left(C_{k_{i_1 j_1}} C_{k_{i_1 j_2}} \cdots \right) \prod_i \sigma_i^{n_i}$$

$$= \sum_{\{k_{ij}\}} \left(C_{k_{i_1 j_1}} C_{k_{i_2 j_2}} \cdots \right) \sum_{\{\sigma_i\}} \left(\sigma_1^{n_1} \sigma_2^{n_2} \cdots \right)$$

where $n_i = \sum_{\langle ij \rangle} k_{ij}$

all the neighbors



$$\Rightarrow Z[N,\beta] = \sum_{\{k_{ij}\}} \left(C_{k_{i_1 j_1}} C_{k_{i_2 j_2}} \cdots \right) \prod_i \left(\sum_{\sigma_i} \sigma_i^{n_i} \right)$$

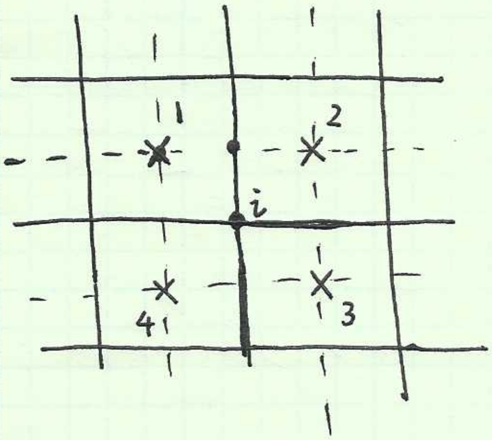
$$= \sum_{\{k_{ij}\}} C_{k_{i_1 j_1}} C_{k_{i_2 j_2}} \prod_i \left[1 + (-)^{n_i} \right]$$

$\Rightarrow$ only the configuration that every site $n_i = \sum_{\langle ij \rangle} k_{ij} =$ even can survive. This is is the constraint

$$\Rightarrow \quad Z[N,\beta] = 2^N \sum_{\{k_{ij}\}}{}' \left(C_{k_{i_1 j_1}} C_{k_{i_2 j_2}} \cdots \right)$$

where $\sum'$ means the constraint $\sum_{\langle ij \rangle} k_{ij} = 0 \pmod 2$

for site i.



Around site i in the original lattice there are four sites in the dual lattice we assign dual spin variable μ_i (Ising) we can solve the constraint as

$k_b = \frac{1}{2}(1 - \mu_{i'}\mu_{j'})$ the bond b intersect with the bond $\langle i'j' \rangle$ in the dual lattice.

$$\Rightarrow \sum_{\text{around } i} k_b = \frac{1}{2}(4 - \mu_1\mu_2 - \mu_2\mu_3 - \mu_3\mu_4 - \mu_4\mu_1)$$

because $\mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_4 + \mu_4\mu_1 = (\mu_1 + \mu_3)(\mu_2 + \mu_4) = \begin{cases} 0 \\ \pm 4 \end{cases}$

$$\Rightarrow \sum_{\text{around } i} k_b \equiv 0 \pmod 2.$$

"One question is that: for every allowed $k_{\langle ij \rangle}$ configuration, does it allways corresponds to a solution of $\mu_{i'}$?" Apparently, flipping the sign of all μ_i', does not change k_b, and thus there's 2 to 1 mapping.

$$\Rightarrow \quad Z = 2^N/2 \sum_{\{\mu\}} C_{k_{i_1 j_1}} C_{k_{i_2 j_2}} \cdots$$

each bond $\langle i_1 j_1 \rangle$ intersect with a bond $\langle i'j' \rangle$ in the dual lattice we denote $C_{k_{i_1 j_1}}$ as $C_{k_{i'j'}}$. And we need to express $C_{k_{i'j'}}$ through $\mu_{i'}$.

$$C_{k_{ij}} = k_{ij} \sinh\beta + (1-k_{ij})\cosh\beta = \frac{1}{2}(1-\mu_{i'}\mu_{j'})\, sh\beta + \frac{1}{2}(1+\mu_{i'}\mu_{j'})\, ch\beta$$

$$= \frac{1}{2} e^{\beta}(1+\mu_{i'}\mu_{j'} e^{-2\beta}) = \frac{e^{\beta}}{2ch\beta^*}[ch\beta^* + \mu_{i'}\mu_{j'}\, sh\beta^*]$$

with $\tanh\beta^* = e^{-2\beta}$

$$\Rightarrow \quad C_{k_{ij}} = \left[2\sinh 2\beta^*\right]^{-1/2} e^{\beta^* \mu_{i'}\mu_{j'}}$$

$$\Rightarrow \quad Z = \frac{1}{2}\left[\sinh 2\beta^*\right]^{-N} \sum_{\{\mu\}} \exp[\beta^* \Sigma \mu_{i'}\mu_{j'}]$$

realation $\boxed{\tanh\beta^* = e^{-2\beta}}$

or $\boxed{\sinh 2\beta \, \sinh 2\beta^* = 1}$