$4-\epsilon$ (IV) Crossover — integrate over RG equations ①

In this lecture, we will study the evolution of the critical exponents from mean field values to the critical vaules. For simplicity, we take $r\Lambda^{-2} \to r$, and $u/\Lambda^{\epsilon} \to u$ in this lecture to simplify notation, i.e. r and u here are already dimensionless. We already have

$$\begin{cases} \dfrac{dr}{d\ln l} = 2r + A\dfrac{u}{1+r} & ① \\ \dfrac{du}{d\ln l} = u\left[\epsilon - \dfrac{Bu}{(1+r)^2}\right] & ② \end{cases}$$

$A = 3K_d$ and $K_4 = \dfrac{2\pi^2}{(2\pi)^d}\Big|_{d=4} = \dfrac{1}{8\pi^2}$

$B = 9K_d$

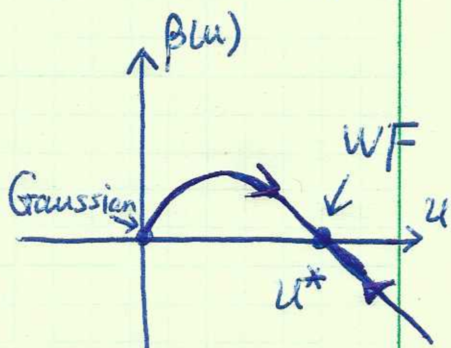
comment: for the $O(n)$ model, the RG equations are similar. Except that A and B are multiplied by a facter as

$$A = (n+2)K_d, \quad B = K_d(n+8).$$

check Chaikin P271 for details

In Eq ②, let's set $r=0$, which is correct at linear order of ϵ,

$$\boxed{\beta(u) = \frac{du}{d\ln l} = u(\epsilon - Bu)}$$



we can integrate this equation

$$u^* = \frac{\epsilon}{B}$$

$$\frac{du}{-Bu(u-u^*)} = d\ln l$$

$$\frac{1}{Bu^*}\left[\frac{1}{u} - \frac{1}{u-u^*}\right]du = d\ln l$$

$$\frac{1}{Bu^*}\ln\frac{u}{u-u^*} = d\ln l + C \Rightarrow \frac{u(l)}{u(l)-u^*} = C\cdot e^{Bu^*\ln l}$$

and $\dfrac{u_0}{u_0-u^*} = C$. u_0 is the initial value at $\ln l = 0$.

$$\frac{u(l)}{u(l)-u^*} = \frac{u_0}{u_0-u^*}\, e^{\epsilon \ln l} \Rightarrow 1-\frac{u^*}{u(l)} = \left(1-\frac{u^*}{u_0}\right) e^{-\epsilon \ln l}$$

$$\boxed{\frac{u(l)}{u_0} = \frac{e^{\epsilon \ln l}}{1+u_0/u^*(e^{\epsilon \ln l}-1)} \to \begin{cases} 1 & \text{as } \ln l \to 0 \\ u^* & \text{as } \ln l \to \infty. \end{cases}}$$

Now we integrate $r(l)$, to correct at the order of ϵ, we write

$$\frac{dr}{d\ln l} = (2-Au(l))r(l) + Au(l) + Au(l)\left[\frac{1}{1+r(l)} - 1 + r(l)\right]$$

Then we set $r(l) = \tilde{r}(l)\, e^{S(l)}$, and $S(l) = \int_0^{\ln l} (2-Au(l'))\, d\ln l'$

since $u(l')$ is $O(\epsilon)$, thus $S(l) = 2\ln l - \int_0^{\ln l} A u(l')\, d\ln l' = 2\ln l + O(\epsilon)$.

Plug in $r(l) = \tilde{r}(l)\, e^{S(l)}$ into the RG equation

$$\Rightarrow \frac{d\tilde{r}(l)}{d\ln l} = e^{-S(l)}\left[A u(l) + Au(l)\left(\frac{1}{1+r(l)} - 1 + r(l)\right)\right]$$

$$\Rightarrow \tilde{r}(l) = r_0 + \int_0^{\ln l} e^{-S(l')}\left[A u(l') + Au(l')\left(\frac{1}{1+r(l)} - 1 + r(l)\right)\right] d\ln l'$$

$\tilde{r}_0 = r_0$

since $du/dl = O(\epsilon^2)$, $e^{-S(l)} = e^{-2\ln l}(1+O(\epsilon))$, $r(l) = e^{2\ln l} r_0 + O(\epsilon)$

$$\Rightarrow \int_0^{\ln l} e^{-S(l')} A u(l')\, d\ln l' \simeq -\frac{A}{2}\int_0^{\ln l} u(l')\, d e^{-S(l')} = -\frac{A}{2} u(l') e^{-S(l')}\Big|_0^{\ln l} + O(\epsilon^2)$$

$$= -\frac{A}{2}\left[e^{-S(l)} u(l) - u(0)\right] + O(\epsilon^2)$$

$$\int_0^{\ln l} d\ln l'\, e^{-S(l')} A u(l')\left[\frac{1}{1+r(l)} - 1 + r(l)\right] = \int_0^{\ln l} d\ln l'\, Au(l')\left[e^{-2\ln l}\frac{r^2}{1+r} + O(\epsilon)\right)$$

$$= \int_0^{\ln l} d\ln l' \; A\, u(l') \left[\frac{r_0\, r(l')}{1+r(l')} + O(\epsilon) \right]$$

$$\frac{r(l')}{1+r(l')} = \frac{1}{2} \frac{d}{d\ln l'} \ln(1+r(l')) + O(\epsilon)$$

$$\Rightarrow \text{The above Eq} = \int_0^{\ln l} d\ln l' \; \frac{r_0}{2} A\, u(l') \left[\frac{d}{d\ln l'} \ln(1+r(l')) + O(\epsilon) \right]$$

$$= \frac{r_0}{2} A\, u(l') \ln(1+r(l')) \Big|_0^{\ln l} + O(\epsilon^2)$$

$$= \frac{A}{2} \left[r_0\, u(l) \ln(1+r(l)) - r_0\, u_0(0) \ln(1+r_0(0)) \right] + O(\epsilon)$$

$$r_0 = e^{-S(l)}\, r(l) + O(\epsilon)$$

$$\Rightarrow \text{The above Eq} = \frac{1}{2} e^{-S(l)} A\, u(l) \underbrace{\ln(1+r(l))}_{r(l)} - \frac{A}{2} u_0\, r_0 \ln(1+r_0)$$

Combine ~~together~~, we have

$$\tilde{r}(l) = r_0 - \frac{A}{2}\left[e^{-S(l)} u(l) - u(0) \right] + \frac{A}{2}\left[e^{-S(l)} u(l)\, r(l) \ln(1+r(l)) - u_0\, r_0 \ln(1+r_0) \right]$$

$$r(l) = \tilde{r}(l)\, e^{S(l)}$$

define $\boxed{t(l) = r(l) + \frac{A}{2} u(l) - \frac{A}{2} u(l)\, r(l) \ln(1+r(l))}$ plugin $r(l)$

we have $t(l) = r_0 e^{S(l)} + \frac{A}{2} u_0 e^{S(l)} - \frac{A}{2} u_0\, r_0 \ln(1+r_0)\, e^{S(l)}$

$$\boxed{t(l) = e^{S(l)}\, t_0 \quad \text{and} \quad S(l) = \int_0^{\ln l} (2 - A\, u(l'))\, d\ln l'}$$

t is the actually physical ~~temperature~~ $t = \frac{T - T_c}{T_c}$

The phase transition occurs at $t=0$,

or the interaction shifts the T_c. For initial condition

if we want right at T_c, we need $\boxed{r_0 = -\frac{A}{2}u_0 + O(\epsilon^2)}$.

Now let us evaluate $S(l)$ to the order of ϵ:

$$S(l) = 2l - A\int_0^{\ln l} d\ln l'\, u(l') = 2l - Au_0 \int_0^{\ln l} d\ln l' \frac{e^{\epsilon \ln l'}}{1 + u_0/u^*(e^{\epsilon \ln l'} - 1)}$$

$$= 2l - \frac{Au_0 u^*}{\epsilon\, u_0}\int \frac{d e^{\epsilon \ln l'}}{e^{\epsilon \ln l'} - 1 + u^*/u_0}$$

$$= 2l - \frac{A}{B}\ln\left[e^{\epsilon \ln l'} - 1 + \frac{u^*}{u_0}\right]\Bigg|_{\ln l' = 0}^{\ln l' = \ln l}$$

$$\boxed{S(l) = 2l - \frac{A}{B}\ln\left[1 + \frac{u_0}{u^*}\left(e^{\epsilon \ln l} - 1\right)\right] + O(\epsilon^2)}$$

$$\Rightarrow \boxed{\frac{t(l)}{t(0)} = e^{2\ln l}\left[Q(l)\right]^{-\frac{A}{B}}\, t(0)}, \text{ where } Q(l) = 1 + \frac{u_0}{u^*}\left[e^{\epsilon \ln l} - 1\right].$$

as $\ln l \to 0$, $\quad t(l) \simeq e^{2\ln l}\, t(0)$

$\ln l \to \infty \quad t(l) \simeq e^{(2 - \frac{A}{B}\epsilon)\ln l}\, \frac{u_0}{u^*}\, t(0)$.

We have interpreted that the power coefficient is ν^{-1}, and thus

away from the critical regime, we have $\nu^{-1} = 2$,

at the critical region, we have $\nu^{-1} = 2 - \frac{A}{B}\epsilon$.

When we do RG from a nonzero $t(0)$, we cannot run forever. When $t(\ell) \sim 1$, it means $r(\ell)\Lambda^2 \sim \Lambda^2$ already beyound the cutoff, and we have to stop. A rought estimation, we can use the mean-field behavior that at

$$2\ln \ell \simeq \ln\left[\frac{1}{t(0)}\right],$$

we need to stop. But in order to exhibit non-Gaussian behavior, we need $\frac{u}{u^*} e^{\epsilon \ln \ell} > 1$ at $\ln \ell \sim \frac{1}{2}\ln\left(\frac{1}{t(0)}\right)$

$$\Rightarrow \quad \frac{u_0}{u^*}\left(\frac{1}{t(0)}\right)^{\frac{\epsilon}{2}} > 1$$

$$\text{i.e} \quad t(0) < \left(\frac{B u_0}{\epsilon}\right)^{2/\epsilon}$$

this is consistent with the Ginzburg criterion.

§ Crossover of magnetic susceptibility

$$\chi = G^{-1}(q=0, r, u).$$

Let us consider along the RG flow $(r(\ell), u(\ell))$, we have already had the relation of correlation function

$$G(R/\ell, r(\ell), u(\ell)) = \ell^{2(d-y_h)}\, G(R, r, u)$$

The Fourier component

$$G(q, r, u) = \int d^dR \; e^{i\vec{q}\cdot\vec{R}}\, G(R, r, u)$$

$$= \ell^{-2(d-y_h)}\, \ell^d \int d^d\frac{R}{\ell}\; e^{i\ell\vec{q}\cdot\vec{R}/\ell}\, G(R/\ell, r(\ell), u(\ell))$$

$$= \ell^{-(d-2+\eta)+d}\; G(\ell q, r(\ell), u(\ell)) = \ell^{2-\eta}\, G(\ell q, r(\ell), u(\ell))$$

i.e. we have the scaling behavior of Green's function

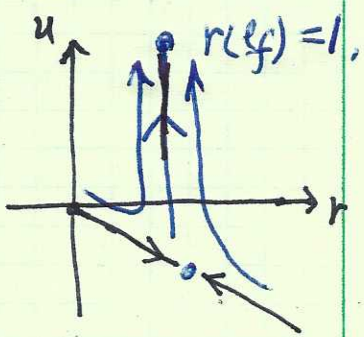
$$\boxed{G(q, r, u) = l^{2-\eta}\, G(lq, r(l), u(l))}$$

Now we set $q=0$, and at one-loop level $\eta = 0 + O(\epsilon^2)$, thus can be neglected. We write down the simplified version

$$G(r_0, u_0) = G(r(l), u(l)) \quad (q \text{ set to } 0)$$

Let use choose the ending point of the RG process at $\ln l_f$, such that $t(l_f) = 1$. This is region where perturbation theory applies. ↙ u_f is small at ϵ.

and we can safely put $G(r(l_f)) = \dfrac{1}{t(l_f)} = 1$.



Then we have $G(r_0, u_0) = l_f^2\, G(r_f, u_f) = l_f^2$

Then what is l_f? It should be detemined by

$$\frac{t(l_f)}{t(0)} = l_f^2 \left(Q(l_f)\right)^{-\frac{A}{B}} = \frac{1}{t(0)}$$

$$\Rightarrow t(0)^{-1/2}\left[1 + \frac{u_0}{u^*}\left(e^{\epsilon \ln l_f} - 1\right)\right]^{\frac{A}{2B}} = e^{\ln l_f}$$

$$\Rightarrow e^{\ln l_f} \simeq t(0)^{-1/2}\left[1 + \frac{u_0}{u^*}\left[\left(\frac{1}{t(0)}\right)^{\epsilon/2} - 1\right]\right]^{\frac{A}{2B}} \quad (\text{iteration})$$

because $t(0) \ll 1$, $\Rightarrow$

$$l_f = e^{\ln l_f} \simeq t(0)^{-1/2}\left[1 + \frac{B}{\epsilon}\frac{u_0}{t(0)^{\epsilon/2}}\right]^{\frac{A}{2B}}$$

$$\boxed{G(r_0, u_0) = t(0)^{-1}\left[1 + \frac{B u_0}{\epsilon\, t(0)^{\epsilon/2}}\right]^{\frac{A}{B}}}$$

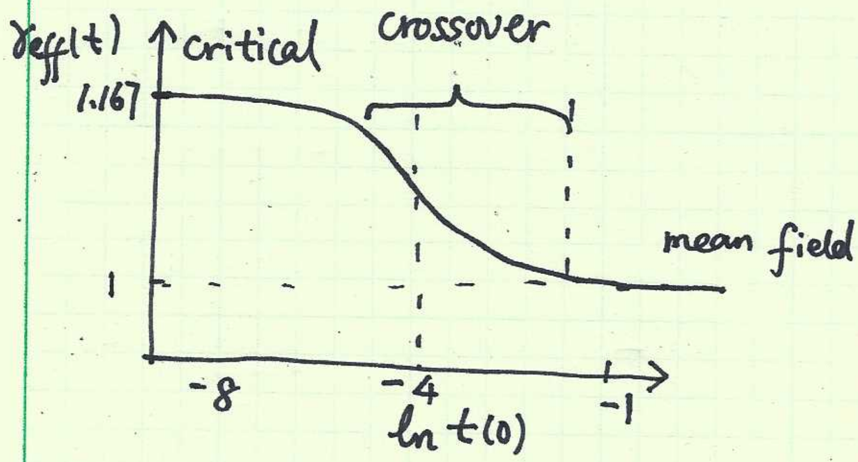
Thus

$$\chi \simeq G = \begin{cases} t(0)^{-1} & \text{if } \dfrac{Bu_0}{\epsilon} \ll t(0)^{\epsilon/2} \qquad \nu = 1 \text{ (Curie-Weiss)} \\[2ex] [t(0)]^{-1-\frac{\epsilon}{2}\left(\frac{A}{B}\right)} & \text{if } \dfrac{Bu_0}{\epsilon} \gg t(0)^{\epsilon/2} \qquad \nu = 1 + \dfrac{\epsilon}{2}\dfrac{A}{B} \end{cases}$$

again it can exhibit two different scaling outside / inside the critical regin whose boundary is determined by the Ginzburg criterion.

We can deffine $\gamma_{eff}(t) = -\dfrac{d \ln G}{d \ln t}$

For $n=1$, $\epsilon = 1$, $Bu_0 = 10^{-2}$, we have



The crossover ~~temperature~~ $t_{cr} \sim (Bu_0)^2 = 10^{-4}$,

and thus the critical regin is narrow compared to the mean field regin $t_{cr} < t < 1$.