Lect 14: SO(5) group and Sp(4)

§ Cliford algebra

① rank-1: $\sigma_1, \sigma_2, \sigma_3$, $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$

② rank-2 $\Gamma^1 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}$, $\Gamma^{2-4} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}$, $\Gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

$\uparrow$ σ_2-type $\quad$ $\uparrow$ σ_3-type $\quad$ $\uparrow$ σ_1-type

They satisfy $\{\Gamma^a, \Gamma^b\} = 2\delta_{ab}$. They can be related to SU(2) spin 3/2 representation $S_z = \begin{pmatrix} 3/2 & & & \\ & 1/2 & & \\ & & -1/2 & \\ & & & 3/2 \end{pmatrix}$ $\quad S_x + iS_y = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$S_x - iS_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$

$\Gamma^1 = \frac{1}{\sqrt{3}}[S_x S_y + S_y S_x]$,

$\Gamma^2 = \frac{1}{\sqrt{3}}[S_x S_z + S_z S_x]$ $\quad \Gamma^4 = S_z^2 - 5/4$

$\Gamma^3 = \frac{1}{\sqrt{3}}[S_z S_y + S_y S_z]$ $\quad \Gamma^5 = \frac{1}{\sqrt{3}}[S_x^2 - S_y^2]$

In other words, Γ^a $(a = 1 \sim 5)$ can be viewed as spin gudrapole matrices

For the rank-1 case, there are no other degrees of freedom: The product of σ's remains an σ. Now the product of Γ is not a Γ any more

we define
$$\boxed{\Gamma^{ab} = -\frac{i}{2}[\Gamma^a, \Gamma^b] = -i\Gamma^a\Gamma^b \quad (a<b)}$$

There are 10 Γ^{ab}'s. Hence $5\,\Gamma^a + 10\,\Gamma^{ab}$ = complete set of 4×4 traceless Hermitian matrices.

HW: prove that

$$[\Gamma^{ab}, \Gamma^{c}] = 2i\{\delta^{ac}\Gamma^{b} - \delta^{bc}\Gamma^{a}\}$$

$$[\Gamma^{ab}\ \Gamma^{cd}] = 2i\{\delta^{ac}\Gamma^{bd} + \delta^{bd}\Gamma^{ac} - \delta^{ad}\Gamma^{bc} - \delta^{bc}\Gamma^{ad}\}$$

§ SO(N) algebra

Consider N-dimensional $\underset{\text{real}}{\text{space}}$ $x_1, \cdots x_n$, and associated momenta $P_1, \cdots P_n$. We define $L_{ab} = x_a P_b - x_b P_a$. Then we have

$$[L_{ab}, x_c] = [x_a P_b - x_b P_a, x_c] = x_a(-i)\delta_{bc} - x_b(-i)\delta_{ac}$$

$$= i[\delta_{ac} x_b - \delta_{bc} x_a]$$

$$[L_{ab}, L_{cd}] = [x_a P_b - x_b P_a, x_c P_d - x_d P_c]$$

According to $[x_i P_j, x_e P_m] = x_i [P_j, x_e] P_m - x_e [P_m, x_i] P_j$

$$= i x_e P_j \delta_{im} - i x_i P_m \delta_{je}$$

$$\rightarrow [L_{ab}, L_{cd}] = i[\delta_{ad} x_c P_b - i\delta_{bc} x_a P_d + \delta_{bc} x_d P_a - i\delta_{ad} x_b P_c] + i[\delta_{ac} x_b P_d - \delta_{bd} x_c P_a + \delta_{bd} x_a P_c - \delta_{ac} x_d P_b]$$

$$= i[-\delta_{ad}(x_b P_c - x_c P_b) - \delta_{bc}(x_a P_d - x_d P_a) - \delta_{ac}(x_d P_b - x_b P_d) - \delta_{bd}(x_c P_a - x_a P_c)]$$

$$= i[\delta_{ac} L_{bd} + \delta_{bd} L_{ac} - \delta_{ad} L_{bc} - \delta_{bc} L_{ad})$$

Hence; we use spin 3/2 spinor $\psi = \begin{bmatrix} C_{3/2} \\ C_{1/2} \\ C_{-1/2} \\ C_{-3/2} \end{bmatrix}$. then

$$\boxed{\begin{aligned} n^a &= \tfrac{1}{2}\psi^\dagger \Gamma^a \psi \\ L^{ab} &= \tfrac{1}{2}\psi^\dagger \Gamma^{ab} \psi \end{aligned}}$$

transforms like SO(5) 5-vector, and 10-generator.

Ⓧ Analogy to $SO(3)/SU(2)$

Consider a spin-½ spinor $\psi = \begin{pmatrix} \psi_{1/2} \\ \psi_{-1/2} \end{pmatrix}$. We define an $SU(2)$ rotation $|\psi^g\rangle = \mathcal{U}(g)|\psi\rangle$ with $\boxed{\mathcal{U}(g) = e^{-i\vec{S}\cdot\hat{n}\theta}}$ where $g(\hat{n},\theta)$ is a rotation around $\hat{n}$-axis at the angle of θ. Then

$$\boxed{\vec{r}^{\,g} = g(n,\theta)\,\vec{r}, \qquad \vec{S}^{\,g} = g(n,\theta)\,\vec{S}}$$

where here $g(n,\theta)$ is intepreted as a 3×3 orthogonal matrix. Then

$$\langle \psi^g | S_\mu | \psi^g \rangle = g_{\mu\nu} \langle \psi | S_\nu | \psi \rangle \Rightarrow \boxed{\mathcal{U}^\dagger(g)\, S_\mu\, \mathcal{U}(g) = g_{\mu\nu}\, S_\nu}$$

Now we have a spin-3/2 spinor $\psi = \begin{pmatrix} \psi_{3/2} \\ \psi_{1/2} \\ \psi_{-1/2} \\ \psi_{-3/2} \end{pmatrix}$, the corresponding $Sp(4)$ transformation

$$\boxed{\begin{array}{l} |\psi^g\rangle = \mathcal{U}(g)|\psi\rangle, \text{ where } \mathcal{U}(g) = \exp[-i L_{ab}\theta_{ab}] \\ \text{with } L_{ab} = \tfrac{1}{2}\psi^\dagger \Gamma^{ab} \psi \end{array}}$$

Then $\langle \psi^g | n^a | \psi^g \rangle = g_{ab} \langle \psi | n^b | \psi \rangle$, where

g_{ab} here is a 5×5 orthogonal matrix

i.e. $$\boxed{\mathcal{U}^\dagger(g)\; n^a\; \mathcal{U}(g) = g_{ab}\, n_b}$$

mapping between $Sp(4)$ and $SO(5)$.

⊛ Hopf map

① First Hopf

$$S_a = z^*_\alpha \sigma^a_{\alpha\beta} z_\beta \,, \text{ where } z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \text{ satisfying } |z_1|^2 + |z_2|^2 = 1$$

$$(\mathrm{Re}\, z_1)^2 + (\mathrm{Im}\, z_1)^2 + (\mathrm{Re}\, z_2)^2 + (\mathrm{Im}\, z_2)^2 = 1 \longrightarrow z \in S^3 \text{ sphere.}$$

$$S_a \cdot S_a = z^*_\alpha \sigma^a_{\alpha\beta} z_\beta z^*_\gamma \sigma^a_{\gamma\delta} z_\delta$$

according to $\boxed{\sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta}}$ prove it.

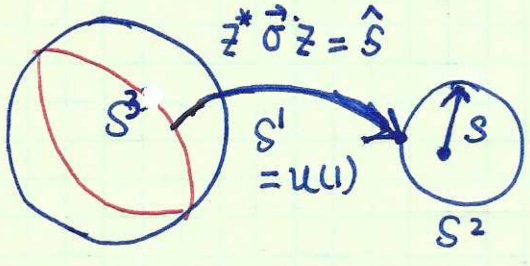
$$\Rightarrow S_a \cdot S_a = z^*_\alpha z^*_\gamma z_\beta z_\delta (2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta}) = (z^*_\alpha z_\alpha)(z_\beta z_\beta) = 1$$

Hence $\vec{S} \in S^2$ Bloch sphere.

We also have $S^a \sigma^a_{\alpha\beta} z_\beta = z^*_\gamma \sigma^a_{\gamma\delta} z_\delta \sigma^a_{\alpha\beta} z_\beta = z^*_\gamma z_\delta z_\beta (2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}$

$$= z_\alpha$$

i.e $(\vec{S}\cdot\vec{\sigma})_{\alpha\beta} z_\beta = z_\alpha$, i.e. z is the eigenstate of $\vec{S}\cdot\vec{\sigma}$ along the the direction of $\vec{S}$.



if we denote $z = \begin{pmatrix} x_1 + i x_2 \\ x_3 + i x_4 \end{pmatrix}$,

then $\begin{pmatrix} [\cos\theta x_1 - \sin\theta x_2] + i[\cos\theta x_2 + \sin\theta \\ [\cos\theta x_3 - \sin\theta x_4] + i[\cos\theta x_4 + \sin\theta x_3 \end{pmatrix}$

$\longrightarrow$ the same $\hat{S} = \begin{pmatrix} 2(x_1x_3 + x_2x_4) \\ 2(x_1x_4 - x_2x_3) \\ (x_1^2 + x_2^2) - (x_3^2 + x_4^2) \end{pmatrix}$ invarient

$$\begin{cases} y_1 = \cos\theta\, x_1 - \sin\theta\, x_2 \\ y_2 = \sin\theta\, x_1 + \cos\theta\, x_2 \\ y_3 = \cos\theta\, x_3 - \sin\theta\, x_4 \\ y_4 = \sin\theta\, x_3 + \cos\theta\, x_4 \end{cases}$$

$\rightarrow$ big circle on the S^3 sphere since if y on the cricle, so does $-y$.

② ~~2nd Hopf map~~

$$x_a = z^*_\alpha \Gamma^a z_\beta, \quad \text{where } z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \in S^7, \text{ i.e. } |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1.$$

We need some identities of Clifford algebra

HW: prove

$$\delta_{\alpha\delta}\,\delta_{\gamma\beta} = \tfrac{1}{4}\left[\delta_{\alpha\beta}\delta_{\gamma\delta} + \Gamma^a_{\alpha\beta}\Gamma^a_{\gamma\delta} + \Gamma^{ab}_{\alpha\beta}\Gamma^{ab}_{\gamma\delta}\right]$$

$$\Gamma^a_{\alpha\delta}\Gamma^a_{\gamma\beta} = \left[\tfrac{5}{4}\delta_{\alpha\beta}\delta_{\gamma\delta} - \tfrac{3}{4}\Gamma^a_{\alpha\beta}\Gamma^a_{\gamma\delta} + \tfrac{1}{4}\Gamma^{ab}_{\alpha\beta}\Gamma^{ab}_{\gamma\delta}\right]$$

$$\Gamma^{ab}_{\alpha\delta}\Gamma^{ab}_{\gamma\beta} = \left[\tfrac{10}{4}\delta_{\alpha\beta}\delta_{\gamma\delta} + \tfrac{2}{4}\Gamma^a_{\alpha\beta}\Gamma^a_{\gamma\delta} - \tfrac{2}{4}\Gamma^{ab}_{\alpha\beta}\Gamma^{ab}_{\gamma\delta}\right.$$

$$\Rightarrow \quad \Gamma^a_{\alpha\beta}\Gamma^a_{\gamma\delta} = \delta_{\alpha\delta}\delta_{\beta\gamma} - \Gamma^a_{\alpha\delta}\Gamma^a_{\gamma\beta} + \delta_{\alpha\beta}\delta_{\gamma\delta}$$

hence $$x_a x_a = z^*_\alpha z^*_\gamma z_\beta z_\delta\left[\Gamma^a_{\alpha\beta}\Gamma^a_{\gamma\delta}\right] = z^*_\alpha z^*_\gamma z_\beta z_\delta\left[\delta_{\alpha\delta}\delta_{\beta\gamma} + \delta_{\alpha\beta}\delta_{\gamma\delta} - \Gamma^a_{\alpha\delta}\Gamma^a_{\gamma\beta}\right]$$

$$= |z|^2|z|^2 + |z|^2|z|^2 - (z^*\Gamma^a z)(z^*\Gamma^a z) \Rightarrow 2x_a x_a = 2 \Rightarrow x_a x_a = 1$$

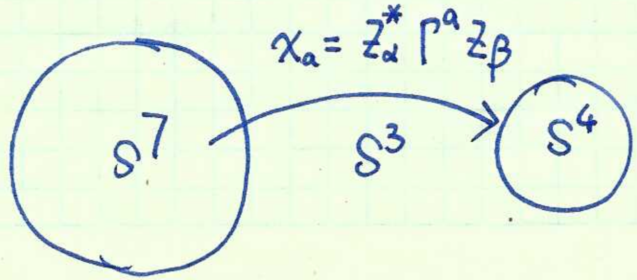
hence $x_a \in S^4$ sphere.

We also check $$(x^a\Gamma^a)_{\alpha\beta} z_\beta = z^*_\gamma \Gamma^a_{\gamma\delta} z_\delta\, \Gamma^a_{\alpha\beta} z_\beta = z^*_\gamma z_\delta z_\beta\left[\delta_{\alpha\delta}\delta_{\beta\gamma} + \delta_{\alpha\beta}\delta_{\gamma\delta} - \Gamma^a_{\alpha\delta}\Gamma^a_{\gamma\beta}\right]$$

$$= z_\alpha + z_\alpha - (z^*_\gamma \Gamma^a_{\gamma\beta} z_\beta)\,\Gamma^a_{\alpha\sigma} z_\sigma$$

$$\Rightarrow 2(x^a\Gamma^a)_{\alpha\beta} z_\beta = 2z_\alpha \quad \text{or} \quad \boxed{x^a\Gamma^a z = z},$$

i.e z is the eigenstate of $x^a \cdot \Gamma^a$ with the positive eigenvalue 1.



For a given $x^a \in S^4$, how to parameterize z ?

$$P = \frac{1 + x^a \Gamma^a}{2} \quad \text{and} \quad z = P \begin{pmatrix} u \\ 0 \end{pmatrix} \quad \text{where } u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with $|u_1|^2 + |u_2|^2 = 1$.

For simplicity, we take here $\Gamma^{1-3} = \begin{pmatrix} 0 & +i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix}$, $\Gamma^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\Gamma^5 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

then $x_i \Gamma_i = \begin{pmatrix} x_5 & x_4 + i\vec{x}\cdot\vec{\sigma} \\ x_4 - i\vec{x}\cdot\vec{\sigma} & -x_5 \end{pmatrix}$

$$\Rightarrow P = \begin{pmatrix} \frac{1+x_5}{2} & \frac{x_4 + i\vec{x}\cdot\vec{\sigma}}{2} \\ \frac{x_4 - i\vec{x}\cdot\vec{\sigma}}{2} & \frac{1-x_5}{2} \end{pmatrix}$$

$$P \begin{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1+x_5}{2} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \frac{(x_4 - i\vec{x}\cdot\vec{\sigma})}{2} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{pmatrix} = \psi$$

$$\langle \psi | \psi \rangle = \left(\frac{1+x_5}{2}\right)^2 + \frac{x_4^2 + x_1^2 + x_2^2 + x_3^2}{4} = \frac{1 + 1 + 2x_5}{4} = \frac{1+x_5}{2}$$

$\Rightarrow$ After normalization

$$z = \begin{pmatrix} \sqrt{\frac{1+x_5}{2}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \sqrt{\frac{1}{2(1+x_5)}} (x_4 - i\vec{x}\cdot\vec{\sigma}) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{pmatrix}.$$

If we use $z = P \begin{pmatrix} 0 \\ u \end{pmatrix}$, we arrive at another gauge that

$$z' = \sqrt{\frac{1}{2(1-x_5)}} \begin{pmatrix} (x_4 + i\vec{x}\cdot\vec{\sigma}) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ (1-x_5) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{pmatrix}.$$

Any $SU(2)$ rotation $\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = W \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ does not change the point on S^4.

(*) Root structure

Generators $\boxed{(L_{ab})_{cd} = -i(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \quad 1 \leq a < b \leq 5}$

Then $[L_{ab}, L_{cd}] = i[\delta_{ac}L_{bd} + \delta_{bd}L_{ac} - \delta_{ad}L_{bc} - \delta_{bc}L_{ad}]$

or, more explicibly: generaters in the fundamental vecter Rep.

$L_{12}: \begin{bmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & & & \\ 0 & 0 & & 0 & \\ 0 & 0 & & & \end{bmatrix}$ $L_{13}: \begin{bmatrix} 0 & 0 & -i & & \\ 0 & 0 & 0 & 0 & \\ i & 0 & 0 & & \\ & 0 & & 0 & \end{bmatrix}$ $L_{14} \begin{bmatrix} 0 & 0 & 0 & -i & 0 \\ 0 & & & & 0 \\ 0 & & 0 & & 0 \\ i & & & & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$L_{15}: \begin{bmatrix} 0 & 0 & 0 & 0 & -i \\ 0 & & & & 0 \\ 0 & & 0 & & 0 \\ 0 & & & & 0 \\ i & 0 & 0 & 0 & 0 \end{bmatrix}$, $L_{23}: \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & & \\ 0 & i & 0 & 0 & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \end{bmatrix}$ $L_{25}: \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \end{bmatrix}$

$L_{34} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $L_{35} \begin{bmatrix} & & 0 & & \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \end{bmatrix}$ $L_{24}: \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,

$L_{45} \begin{bmatrix} & & 0 & & \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{bmatrix}$

(*) Cartan Subalgebra

$H_1 = L_{12}$. $H_3 = L_{34}$, and there are 8 remaining roots.

The adjoint Reprensentation is 10-dimensional

According to the definition of adjoint Rep; if $[T^a T^b] = i f_{abc} T_c$

we have $(T^{ad}_a)_{bc} = -i f_{abc}$.

Now we take $a = 12$

b	c	$(I^{ad}_{12})_{bc}$
13	23	$-i$
23	13	i
14	24	$-i$
24	14	i
15	25	$-i$
25	15	i

$$\mathrm{Tr}[I^{ad}_{12} I^{ad}_{12}] = 6$$

hence

$$\mathrm{Tr}[I^{ad}_{A} I^{ad}_{B}] = 6\delta_{AB}$$

more generally for $SO(N)$ group it's adjonit Rep

$$\mathrm{Tr}[I^{ad}_{A} I^{ad}_{B}] = 2(N-2)\,\delta_{AB}$$

The Casimir of the $SO(N)$ adjoint Rep is $C_{SO(N)}(ad) = 2(N-2)$.

Construction of roots.

$$E_{\pm\alpha_1} = \frac{1}{2}\left(L_{13} + L_{24} \pm i(L_{23} - L_{14})\right) \qquad \alpha_1 = (1, -1)$$

$$E_{\pm\alpha_2} = \frac{1}{\sqrt{2}}[L_{35} \pm i L_{45}] \qquad \alpha_2 = (0, 1)$$

$$E_{\pm\alpha_3} = \frac{1}{2}[L_{13} - L_{24} \pm i(L_{23} + L_{14})] \qquad \alpha_3 = (1\ 1)$$

$$E_{\pm\alpha_4} = \frac{1}{\sqrt{2}}[L_{25} \mp i L_{15}] \qquad \alpha_4 = (1, 0)$$

HW: please check

$$[H_i, E_{\alpha_j}] = \alpha_{j,i} E_{\alpha_j}$$

the ith component of the root vector α_j

$$[E_{\alpha_i}, E_{-\alpha_i}] = \vec{\alpha}_i \cdot \vec{H}, \text{ i.e. } [E_{\alpha_1}, E_{-\alpha_1}] = H_1 - H_2$$

$$[E_{\alpha_2}, E_{-\alpha_2}] = H_2$$

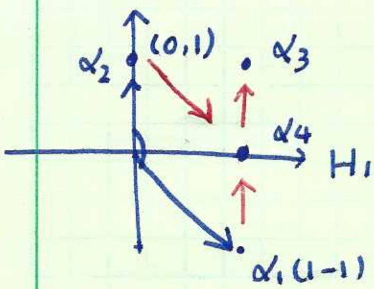
$$[E_{\alpha_3}, E_{-\alpha_3}] = H_1 + H_2$$

$$[E_{\alpha_4}, E_{-\alpha_4}] = H_1 \quad .$$

α_1 and α_2 are simple roots

$\alpha_i, \alpha_i+\alpha_j \cdots \alpha_i+p\alpha_j$ $\quad |\alpha_1|/|\alpha_2| = \sqrt{2}$

angle $= 135°$



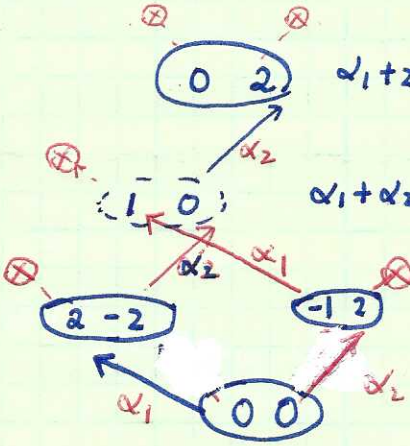
$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

A_{12} start from α_1, apply α_2

Stare from α_2, apply α_1

Construction roots.



$\alpha_1 + 2\alpha_2 = \alpha_3$

$\alpha_1 + \alpha_2 = \alpha_4$

H_i, level 0

(*) Solving for fundamental weights.

$$\frac{2(\alpha_i, \mu_K)}{(\alpha_i, \alpha_i)} = \delta_{iK} \quad \Rightarrow \quad \vec{\mu}_K = \sum_j (A^{-1})_{Kj}\, \vec{\alpha}_j$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$$

$$\vec{\mu}_1 = \frac{1}{2}[2\alpha_1 + 2\alpha_2] = \alpha_1 + \alpha_2 = (1, 0)$$

$$\vec{\mu}_2 = \frac{1}{2}[\alpha_1 + 2\alpha_2] = (1/2, 1/2)$$

Hence, the highese weight $\vec{\mu}^* = \lambda_1 \vec{\mu}_1 + \lambda_2 \vec{\mu}_2 = (\lambda_1 + \frac{\lambda_2}{2}, \frac{\lambda_2}{2})$.

For each representation, ~~we~~ it's characterized by its $\vec{\mu}^*$.

it's dimension $d[\mu^*] = \prod_{\alpha\in\Delta^+}\left(1+\frac{\mu^*\cdot\alpha}{\rho\cdot\alpha}\right)$ where $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$.

Here $\rho = \frac{1}{2}[(1\,-1)+(0,1)+(1,0)+(1,1)] = (\frac{3}{2},\frac{1}{2})$

$$\Rightarrow d(\mu^*) = \left(1+\frac{\lambda_1+\frac{\lambda_2}{2}-\frac{\lambda_2}{2}}{1}\right)\left(1+\frac{\frac{\lambda_2}{2}}{1/2}\right)\left(1+\frac{\lambda_1+\frac{\lambda_2}{2}}{3/2}\right)\left(1+\frac{\lambda_1+\frac{\lambda_2}{2}+\frac{\lambda_2}{2}}{2}\right)$$

$$= (1+\lambda_1)(1+\lambda_2)\left(1+\frac{2\lambda_1+\lambda_2}{3}\right)\left(1+\frac{\lambda_1+\lambda_2}{2}\right) \leftarrow$$ amazing, it's alway an integer.

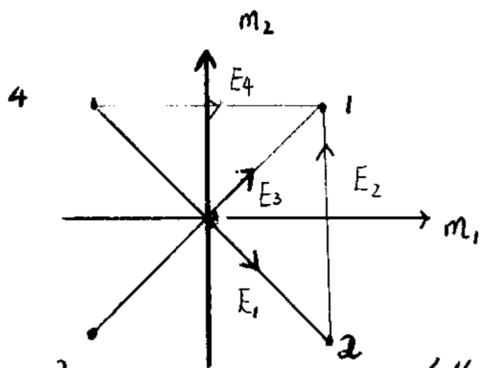
Casimir $C = \vec{\mu}^*\cdot(\vec{\mu}^*+2\rho) = (\lambda_1+\frac{\lambda_2}{2}, \frac{\lambda_2}{2})\cdot(\lambda_1+\frac{\lambda_2}{2}+3, \frac{\lambda_2}{2}+1)$

$$= m_1(m_1+3) + m_2(m_2+1) \leftarrow \text{define} \begin{cases} m_1 = \lambda_1 + \lambda_2/2 \\ m_2 = \lambda_2/2 \end{cases}$$

Some Ir Rep of SO(5) groups

μ_1, μ_2	$M^*(m_1, m_2)$	$d(M^*)$	Casimir $m_1(m_1+3)+m_2(m_2+1)$	Rep
0 0	(0, 0)	1	0	Id
0 1	(½ ½)	4	5/2	spinor
1 0	(1 0)	5	4	vector
0 2	(1, 1)	10	6	ad or antisymmetric tensor
2 0	(2, 0)	14	10	traceless, symmetric tensor
1 1	(3/2 ½)	16	15/2	
1 2	(2 1)	35	12	tensor
0 3	(3/2 3/2)	20	21/2	

a) spinor Rep $M^*(1/2, 1/2)$



$|1\rangle = |1/2, 1/2\rangle$

$|2\rangle = |1/2, -1/2\rangle$

$|3\rangle = |-1/2, -1/2\rangle$

$|4\rangle = |-1/2, 1/2\rangle$

H_1, H_2's matrix is diagonal $\Rightarrow$ $H_1 = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & & -1/2 \end{pmatrix}$ $\quad H_2 = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$

$E_1|4\rangle = a|2\rangle \Rightarrow \langle 4|E_{-1}E_1|4\rangle = \langle 4|[E_{-1}, E_1]|4\rangle = |a|^2 \Rightarrow \Rightarrow |a|^2 = \frac{-1}{6}\langle 4|H_1 - H_2|4\rangle = \frac{1}{6}$

$E_2|2\rangle = a|1\rangle \Rightarrow \langle 2|[E_{-2}, E_2]|2\rangle = \frac{-1}{6}\langle 2|H_2|2\rangle| = |a|^2 \Rightarrow |a|^2 = \sqrt{\frac{1}{12}}$

$E_2|3| = a|4\rangle \Rightarrow \langle 3|[E_{-2}, E_2]|3\rangle = \frac{-1}{6}\langle 3|H_2|3\rangle = |a|^2 \Rightarrow |a|^2 = \sqrt{1/12}$

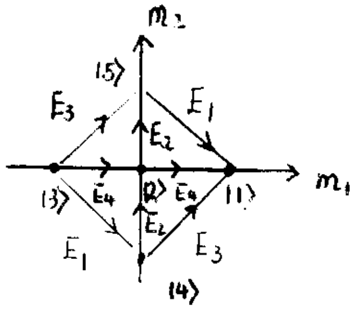
确定位相因子 使之为实数

$E_1 = \sqrt{1/6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad E_2 = \sqrt{1/12}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $\quad E_4 = \sqrt{6}[E_1, E_2] = \sqrt{\frac{1}{12}}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$E_3 = \sqrt{6}\,[E_4, E_2] = \frac{\sqrt{6}}{12}\begin{bmatrix} 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & & 0 \end{bmatrix} = \sqrt{1/6}\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b) vector $\quad M^*(1,0)$ $\qquad |1\rangle = |1\,0\rangle \quad |2\rangle = |00\rangle \quad |3\rangle = |-1,0\rangle$

$|4\rangle = |0,-1\rangle \quad |5\rangle = |0,1\rangle$



$E_1 = a_1 |1\rangle\langle 5| + a_2 |4\rangle\langle 3|$

$E_1 |5\rangle = a_1 |1\rangle \Rightarrow \langle 5| [E_{-1}, E_1] |5\rangle = -\frac{1}{6} \langle 5| H_1 - H_2 |5\rangle = \frac{1}{6} = |a_1|^2$

同理 $|a_2|^2 = 1/6$

$E_2 = b_1 |5\rangle\langle 2| + b_2 |2\rangle\langle 4| \Rightarrow E_2|4\rangle = b_2|2\rangle \Rightarrow \langle 2| E_2 |4\rangle = b_2 \Rightarrow \langle 4| E_2^{\dagger} |2\rangle = b_2^*$

$\langle 4| [E_{-2}, E_2] |4\rangle = \langle 4| -\frac{1}{6} H_2 |4\rangle = \frac{1}{6} = |b_2|^2$;

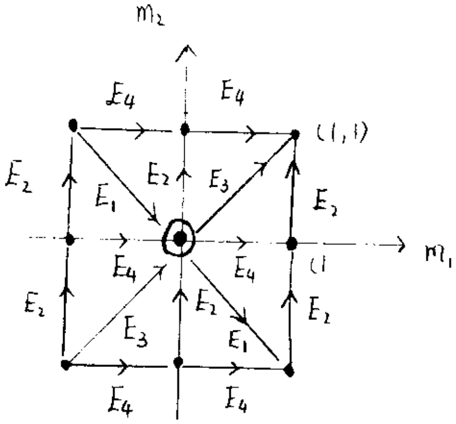
$E_2 |2\rangle = b_1 |5\rangle \Rightarrow \langle 2| E_{-2} E_2 |2\rangle = |b_1|^2 = \langle 2| [E_{-2}, E_2] + E_2 E_{-2} |2\rangle$

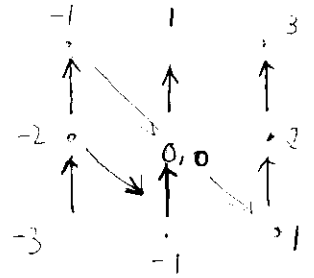
$\Rightarrow \langle 2| -\frac{1}{6} H_2 |2\rangle + |b_2|^2 = |b_1|^2 \Rightarrow |b_1|^2 = |b_2|^2$ 取定 phase factor

$$H_1 = \begin{bmatrix} 1 &&&& \\ & 0 &&& \\ && -1 && \\ &&& 0 & \\ &&&& 0 \end{bmatrix} \quad H_2 = \begin{bmatrix} 0 &&&& \\ & 0 &&& \\ && 0 && \\ &&& -1 & \\ &&&& 1 \end{bmatrix} \quad E_1 = \sqrt{\tfrac{1}{6}} \begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&0 \end{bmatrix} \quad E_2 = \sqrt{\tfrac{1}{6}} \begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&1&0&0&0 \end{bmatrix}$$

$$E_4 = \sqrt{6}\,[E_1, E_2] = \sqrt{\tfrac{1}{6}} \begin{pmatrix} 0&1&0&0&0 \\ 0&0&-1&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} \qquad E_3 = \sqrt{6}\,[E_4, E_2] = \sqrt{\tfrac{1}{6}} \begin{bmatrix} 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&-1&0&0 \end{bmatrix}$$

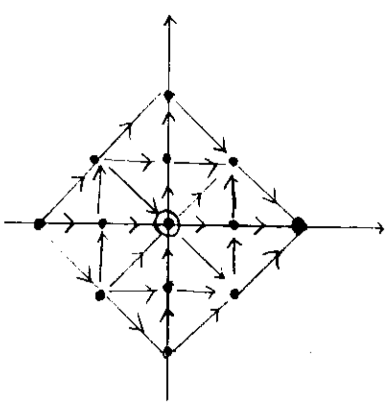
c) tensor (antisymmetric) $M^* = (1, 1)$

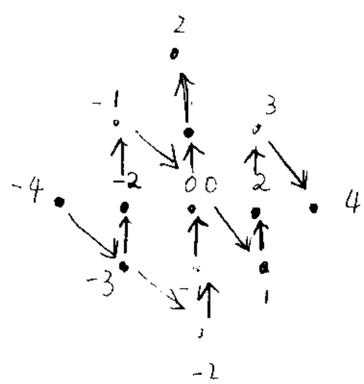


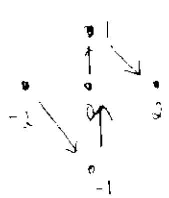


1 + 3

d) tensor (symmetric, traceless tensor) $M^* = (2, 0)$







Rotation of state wavevector and spin operator

① Quantum mechanical level

Rotation in 3d space $g(n,\theta)$: 3×3 orthogonal matrix

$$\vec{r} \to \boxed{\vec{r}\,' = g(n,\theta)\,\vec{r}}\,, \quad \text{or} \quad r'_\mu = g_{\mu\nu}(n,\theta)\, r_\nu$$

$$\vec{S} \to \boxed{\vec{S}\,' = g(n,\theta)\,\vec{S}} \quad \text{or} \quad S'_\mu = g_{\mu\nu}(n,\theta)\, S_\nu.$$

realization at the spinor level

$$|\psi\rangle \to \boxed{|\psi^g\rangle = D(g)\,|\psi\rangle},$$ we should have

$$\langle \psi^g |\, S_\mu \,| \psi^g \rangle = g_{\mu\nu}(n,\theta)\, \langle \psi |\, S_\nu \,| \psi \rangle$$

$$\Rightarrow \boxed{D^\dagger(g)\, S_\mu\, D(g) = g_{\mu\nu}(n,\theta)\, S_\nu}$$

This requirement is the starting point to derive

$$\boxed{D(g) = e^{-i\vec{S}\cdot\hat{n}\theta}}\,.$$

There three boxes are consistent at QM level. (See 杨泽森 or. my QM teaching notes).

② We need to extend this formalism to the case of 2nd quantization.

recall $$e^{-i\vec{S}\cdot\hat{n}\theta}|\alpha\rangle = \sum_\beta |\beta\rangle\langle\beta| e^{-i\vec{S}\cdot\hat{n}\theta}|\alpha\rangle = \sum_\beta |\beta\rangle D_{\beta\alpha}(g)$$

$|\alpha\rangle = c_\alpha^\dagger|0\rangle$ where $|0\rangle$ is rotationally invariant.

$$\Rightarrow \quad e^{-i\vec{S}\cdot\hat{n}\theta}|\alpha\rangle = \underbrace{e^{-i\vec{S}\cdot\hat{n}\theta}\, c_\alpha^\dagger\, e^{i\vec{S}\cdot\hat{n}\theta}}\,|0\rangle = \sum_\beta \underbrace{c_\beta^\dagger\, D_{\beta\alpha}(g)}\,|0\rangle$$

$$\rightarrow \quad e^{-i\vec{S}\cdot\hat{n}\theta}\, c_\alpha^\dagger\, e^{i\vec{S}\cdot\hat{n}\theta} = \sum_\beta c_\beta^\dagger\, D_{\beta\alpha}(g)$$

or $$e^{i\vec{S}\cdot\hat{n}\theta}\, c_\alpha^\dagger\, e^{-i\vec{S}\cdot\hat{n}\theta} = \sum_\beta c_\beta^\dagger\, D_{\beta\alpha}(g^{-1}) = \sum_\beta c_\beta^\dagger\, D^\dagger_{\beta\alpha}(g)$$

$$e^{i\vec{S}\cdot\hat{n}\theta}\, c_\alpha\, e^{-i\vec{S}\cdot\hat{n}\theta} = \sum_\beta c_\beta \left(D^\dagger_{\beta\alpha}(g)\right)^* = \sum_\beta D_{\alpha\beta}(g)\, c_\beta$$

Thus
$$\boxed{\begin{aligned} D^\dagger(g)\, c_\alpha^\dagger\, D(g) &= \sum_\beta c_\beta^\dagger\, D^\dagger_{\beta\alpha}(g) \\ D^\dagger(g)\, c_\alpha\, D(g) &= \sum_\beta D_{\alpha\beta}\, c_\beta \end{aligned}}$$

then $$S_\mu = \tfrac{1}{2} c_\alpha^\dagger\, \sigma^\mu_{\alpha\beta}\, c_\beta \quad \longrightarrow$$

$$S'_\mu = \tfrac{1}{2}\left\{D^\dagger(g)\, c_\alpha^\dagger\, D(g)\right\} \sigma^\mu_{\alpha\beta} \left\{D^\dagger(g)\, c_\beta\, D(g)\right\}$$

$$= \tfrac{1}{2}\, c_\gamma^\dagger\, D^\dagger_{\gamma\alpha}(g)\, \sigma^\mu_{\alpha\beta}\, D_{\beta\delta}(g)\, c_\delta = g_{\mu\nu} S_\nu$$

$$\boxed{D^\dagger_{\gamma\alpha}(g)\, \sigma^\mu_{\alpha\beta}\, D_{\beta\delta}(g) = g_{\mu\nu}\, \sigma^\nu_{\gamma\delta}} \longleftarrow$$

consistent with $D_{\alpha\beta}(g) = e^{-i\frac{\vec{\sigma}}{2}\cdot\hat{n}\theta}$

$C_\alpha^\dagger$ and C_α transforms differently. In order to study pairing, we define $i\sigma_2 C^\dagger$ or $i\sigma_{2,\alpha\beta} C_\beta^\dagger$. we prove that it transforms the same as C_α.

$$i\sigma_{2,\alpha\beta} C_\beta^\dagger \longrightarrow D^\dagger(g)\, i\sigma_{2,\alpha\beta} C_\beta^\dagger D(g) = i\sigma_{2,\alpha\beta}\, C_\gamma^\dagger\, D^\dagger_{\gamma\beta}(g)$$

$$= C_\gamma^\dagger \underbrace{i\sigma_{2,\alpha\beta}\, D^\dagger_{\gamma\beta}(g)}$$

$$i\sigma_{2,\alpha\beta}\, D^\dagger_{\gamma\beta}(g) = i\sigma_{2,\alpha\beta}\, D^*_{\beta\gamma}(g) = i\,[\sigma_2\, D^*(g)]_{\alpha\gamma}$$

$$\boxed{\sigma_2\, \vec{\sigma}\, \sigma_2 = -\vec{\sigma}^*} \quad \text{or} \quad \sigma_2\, \vec{\sigma}^*\, \sigma_2 = -\vec{\sigma}$$

$$\Rightarrow \boxed{\sigma_2\, e^{i\frac{\vec{\sigma}^*}{2}\cdot\hat{n}\theta}\, \sigma_2 = e^{-i\frac{\vec{\sigma}}{2}\cdot\hat{n}\theta}}$$

$$\Rightarrow i\sigma_{2,\alpha\beta}\, D^\dagger_{\gamma\beta}(g) = i[\sigma_2 D^*(g)]_{\alpha\gamma} = \left[e^{-i\frac{\vec{\sigma}}{2}\cdot\hat{n}\theta}\, i\sigma_2\right]_{\alpha\gamma} = (D(g)\, i\sigma_2)_{\alpha\gamma}$$

$$\Rightarrow \boxed{D^\dagger(g)\, [\underbrace{i\sigma_2 C^\dagger}]_\alpha\, D(g) = \sum_\beta D_{\alpha\beta}(g) (i\sigma_2 C^\dagger)_\beta}$$

Similarly

$$\boxed{D^\dagger(g)\, [C(-i\sigma_2)]_\alpha\, D(g) = \sum_\beta \underbrace{[C(-i\sigma_2)]_\beta}\, D^\dagger_{\beta\alpha}(g)}$$

in other words $(i\sigma_2 C^\dagger)_\alpha = i\sigma_{2,\alpha\beta} C_\beta^\dagger \xrightarrow{\text{transf as}} C_\alpha$

$$[C(-i\sigma_2)]_\alpha = C_\beta (-i\sigma_2)_{\beta\alpha} \longrightarrow C_\alpha^\dagger.$$