Lect 15   SO(6) (SU(4)).

Let us denote Γ_a as Γ_{a6}, then together with $\Gamma_{ab} = -i\Gamma_a\Gamma_b$ $(1 \le a < b \le 5)$,

we have

$$[\Gamma_{ab}, \Gamma_{cd}] = 2i[\delta_{ac}\Gamma_{bd} + \delta_{bd}\Gamma_{ac} - \delta_{ad}\Gamma_{bc} - \delta_{bc}\Gamma_{ad}]$$

For $1 \le a < b \le 6$.   Then $L^{ab} = \frac{1}{2}\psi^\dagger \Gamma^{ab} \psi$ form the generator of

the SO(6) group. we have

$$\Gamma^{12} = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} \quad \Gamma^{13} = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix} \quad \Gamma^{14} = \begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix} \quad \Gamma^{15} = \begin{bmatrix} -1 & \\ & I \end{bmatrix} \quad \Gamma^{16} = \begin{bmatrix} 0 & -iI \\ iI & 0 \end{bmatrix}$$

$$\Gamma^{23} = \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix} \quad \Gamma^{24} = \begin{bmatrix} -\sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix} \quad \Gamma^{25} = \begin{bmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{bmatrix} \quad \Gamma^{26} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{bmatrix}$$

$$\Gamma^{34} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix} \quad \Gamma^{35} = \begin{bmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{bmatrix} \quad \Gamma^{36} = \begin{bmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix}$$

$$\Gamma^{45} = \begin{bmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{bmatrix} \quad \Gamma^{46} = \begin{bmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix}$$

$$\Gamma^{56} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

Apparently, these 15 Γ-matrices form a complete set of the 4×4

Hermitian matrices, or, the basis of SU(4) generators. This is

the equivalence between the SU(4) and SO(6) Lie algebra.

In terms of 4-component spinor $\psi = \begin{pmatrix} c_{3/2} \\ c_{1/2} \\ c_{-1/2} \\ c_{-3/2} \end{pmatrix}$,   we can write down

$L_{ab} = \frac{1}{2}\psi^\dagger \Gamma_{ab} \psi$ explicitly.

Define

$$\Pi_x^\dagger = \psi^\dagger \begin{pmatrix} 0 & \sigma_1 \\ 0 & 0 \end{pmatrix} \psi = C_{3/2}^\dagger C_{-1/2} + C_{1/2}^\dagger C_{-1/2}^-$$

$$\Pi_x = \psi^\dagger \begin{pmatrix} 0 & 0 \\ \sigma_1 & 0 \end{pmatrix} \psi = C_{-3/2}^\dagger C_{3/2} + C_{-1/2}^\dagger C_{1/2}$$

$$\Rightarrow \boxed{\begin{aligned} L_{12} &= \tfrac{1}{2}(\Pi_x^\dagger + \Pi_x) = \mathrm{Re}\,\Pi_x = \tfrac{1}{2}\left[C_{3/2}^\dagger C_{-3/2} + C_{1/2}^\dagger C_{-1/2} + C_{-1/2}^\dagger C_{1/2} + C_{-3/2}^\dagger C_{3/2} \right] \\[4pt] L_{25} &= \tfrac{-i}{2}(\Pi_x^\dagger - \Pi_x) = \mathrm{Im}\,\Pi_x = \tfrac{-i}{2}\left[C_{3/2}^\dagger C_{-3/2} + C_{1/2}^\dagger C_{-1/2} - C_{-1/2}^\dagger C_{1/2} - C_{-3/2}^\dagger C_{3/2} \right] \end{aligned}}$$

Similarly,

$$\Pi_y^\dagger = \psi^\dagger \begin{pmatrix} 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} \psi = -i\left[C_{3/2}^\dagger C_{-3/2} - C_{1/2}^\dagger C_{-1/2} \right]$$

$$\Pi_y = \psi^\dagger \begin{pmatrix} 0 & 0 \\ \sigma_2 & 0 \end{pmatrix} \psi = i\left[C_{-3/2}^\dagger C_{3/2} - C_{-1/2}^\dagger C_{1/2} \right]$$

$$\boxed{\begin{aligned} L_{13} &= \tfrac{1}{2}(\Pi_y^\dagger + \Pi_y) = \mathrm{Re}\,\Pi_y = \tfrac{-i}{2}\left[C_{3/2}^\dagger C_{-3/2} - C_{1/2}^\dagger C_{-1/2} + C_{-1/2}^\dagger C_{1/2} - C_{-3/2}^\dagger C_{3/2} \right] \\[4pt] L_{35} &= \tfrac{-i}{2}(\Pi_y^\dagger - \Pi_y) = \mathrm{Im}\,\Pi_y = -\tfrac{1}{2}\left[C_{3/2}^\dagger C_{-3/2} - C_{1/2}^\dagger C_{-1/2} - C_{-1/2}^\dagger C_{1/2} + C_{-3/2}^\dagger C_{3/2} \right] \end{aligned}}$$

and

$$\Pi_y^\dagger = \psi^\dagger \begin{pmatrix} 0 & \sigma_3 \\ 0 & 0 \end{pmatrix} \psi = \psi_{3/2}^\dagger \psi_{-1/2} + \psi_{1/2}^\dagger \psi_{-3/2}$$

$$\Pi_y = \psi^\dagger \begin{pmatrix} 0 & 0 \\ \sigma_3 & 0 \end{pmatrix} \psi = \psi_{-1/2}^\dagger \psi_{3/2} + \psi_{-3/2}^\dagger \psi_{1/2}$$

[I $\overset{C_\sigma \text{ and } \psi_\sigma \text{ here}}{\underset{\text{may switch}}{\quad}}$]

$$\boxed{\begin{aligned} L_{14} &= \tfrac{1}{2}(\Pi_z^\dagger + \Pi_z) = \mathrm{Re}\,\Pi_z = \tfrac{1}{2}\left[\psi_{3/2}^\dagger \psi_{-1/2} + \psi_{1/2}^\dagger \psi_{-3/2} + \psi_{-1/2}^\dagger \psi_{3/2} + \psi_{-3/2}^\dagger \psi_{1/2} \right] \\[4pt] L_{45} &= \tfrac{-i}{2}(\Pi_z^\dagger - \Pi_z) = \mathrm{Im}\,\Pi_z = \tfrac{-i}{2}\left[\psi_{3/2}^\dagger \psi_{-1/2} + \psi_{1/2}^\dagger \psi_{-3/2} - \psi_{-1/2}^\dagger \psi_{3/2} - \psi_{-3/2}^\dagger \psi_{1/2} \right] \end{aligned}}$$

and

$$\boxed{\begin{aligned} L_{23} &= \tfrac{1}{2}\psi^\dagger \begin{pmatrix} \sigma_3 & \\ & \sigma_3 \end{pmatrix} \psi = \tfrac{1}{2}\left[\psi_{3/2}^\dagger \psi_{3/2} - \psi_{1/2}^\dagger \psi_{1/2} + \psi_{-1/2}^\dagger \psi_{-1/2} - \psi_{-3/2}^\dagger \psi_{-3/2} \right] = S_z \\[4pt] L_{24} &= -\tfrac{1}{2}\psi^\dagger \begin{pmatrix} \sigma_2 & \\ & \sigma_2 \end{pmatrix} \psi = \tfrac{i}{2}\left[\psi_{3/2}^\dagger \psi_{1/2} - \psi_{1/2}^\dagger \psi_{3/2} + \psi_{-1/2}^\dagger \psi_{-3/2} - \psi_{-3/2}^\dagger \psi_{-1/2} \right] = -S_y \\[4pt] L_{34} &= \tfrac{1}{2}\psi^\dagger \begin{pmatrix} \sigma_1 & \\ & \sigma_1 \end{pmatrix} \psi = \tfrac{1}{2}\left[\psi_{3/2}^\dagger \psi_{1/2} + \psi_{1/2}^\dagger \psi_{3/2} + \psi_{-1/2}^\dagger \psi_{-3/2} + \psi_{-3/2}^\dagger \psi_{-1/2} \right] = S_x \end{aligned}}$$

$$Q = \frac{\psi^\dagger}{2}\begin{pmatrix} +I & 0 \\ 0 & -I \end{pmatrix}\psi = \tfrac{1}{2}[\psi_{3/2}^\dagger\psi_{3/2} + \psi_{1/2}^\dagger\psi_{+1/2} - \psi_{-1/2}^\dagger\psi_{-1/2} - \psi_{-3/2}^\dagger\psi_{-3/2}]$$

$$L_{15} = -Q$$

$$\frac{\Gamma^{56}+i\Gamma^{16}}{2} = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \Rightarrow \psi_{3/2}^\dagger\psi_{1/2} + \psi_{1/2}^\dagger\psi_{3/2}$$

$$\frac{\Gamma^{56}-i\Gamma^{16}}{2} = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \Rightarrow \psi_{-1/2}^\dagger\psi_{3/2} + \psi_{-3/2}^\dagger\psi_{1/2}$$

$$\Rightarrow n_1 = \frac{-i}{2}[\psi_{3/2}^\dagger\psi_{1/2} + \psi_{1/2}^\dagger\psi_{3/2} - \psi_{-1/2}^\dagger\psi_{3/2} - \psi_{-3/2}^\dagger\psi_{1/2}]$$

$$n_5 = \tfrac{1}{2}[\psi_{3/2}^\dagger\psi_{1/2} + \psi_{1/2}^\dagger\psi_{3/2} + \psi_{-1/2}^\dagger\psi_{3/2} + \psi_{-3/2}^\dagger\psi_{1/2}]$$

$$n_{2,3,4} = \tfrac{1}{2}\left[(\psi_{3/2}^\dagger\,\psi_{1/2}^\dagger)\,\vec{\sigma}\begin{pmatrix}\psi_{3/2}\\\psi_{1/2}\end{pmatrix} - (\psi_{-1/2}^\dagger\,\psi_{-3/2}^\dagger)\,\vec{\sigma}\begin{pmatrix}\psi_{1/2}\\\psi_{-3/2}\end{pmatrix}\right]$$

	1	2	3	4	5	6
1	0	$\mathrm{Re}\,\Pi_x$	$\mathrm{Re}\,\Pi_y$	$\mathrm{Re}\,\Pi_z$	$-Q$	n_1
2		0	S_z	$-S_y$	$\mathrm{Im}\,\Pi_x$	n_2
3			0	S_x	$\mathrm{Im}\,\Pi_y$	n_3
4				0	$\mathrm{Im}\,\Pi_z$	n_4
5					0	n_5
6						0

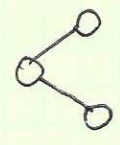$$L_{ab} = \tfrac{1}{2}\psi^\dagger\,\Gamma_{ab}\,\psi$$

We define the Cartan subalgebra (H_1, H_2, H_3) as (L_{51}, L_{23}, L_{46})

or (Q, S_z, n_4). — $Tr[I_A^{ad} I_B^{ad}] = 2(N-2)\delta_{AB} = 8\delta_{AB}$

$$Q = \frac{1}{2}[n_{3/2} + n_{1/2} - n_{-1/2} - n_{-3/2}]$$

$$S_z = \frac{1}{2}[n_{3/2} - n_{1/2} + n_{-1/2} - n_{-3/2}]$$

$$n_4 = \frac{1}{2}\{n_{3/2} - n_{1/2} - n_{-1/2} + n_{-3/2}\}$$

Roots   (Simple roots α_1, α_2 and α_3)

$$\pm\alpha_1 = \pm(1\ -1\ 0) \qquad E_{\alpha_1} = C_{1/2}^\dagger C_{-1/2}, \quad E_{-\alpha_1} = C_{-1/2}^\dagger C_{1/2}$$

$$\pm\alpha_2 = \pm(0\ 1\ -1) \qquad E_{\alpha_2} = C_{-1/2}^\dagger C_{-3/2} \quad E_{-\alpha_2} = C_{-3/2}^\dagger C_{-1/2}$$

$$\pm\alpha_3 = \pm(0,1\ 1) \qquad E_{\alpha_3} = C_{3/2}^\dagger C_{1/2} \quad E_{-\alpha_3} = C_{1/2}^\dagger C_{3/2}$$

$$\pm\alpha_4 = \pm(1,0,1) \qquad E_{\alpha_4} = C_{3/2}^\dagger C_{-1/2} \quad E_{-\alpha_4} = C_{-1/2}^\dagger C_{3/2}$$

$$\pm\alpha_5 = \pm(1,1,0) \qquad E_{\alpha_5} = C_{3/2}^\dagger C_{-3/2} \quad E_{-\alpha_5} = C_{-3/2}^\dagger C_{3/2}$$

$$\pm\alpha_6 = \pm(1,0,-1) \qquad E_{\alpha_6} = C_{1/2}^\dagger C_{-3/2} \quad E_{-\alpha_6} = C_{-3/2}^\dagger C_{1/2}$$

Cartan matrix

$$A_{ij} = \frac{2(\alpha_i \cdot \alpha_j)}{(\alpha_j, \alpha_j)} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3/2 & 1/2 \\ 1 & 1/2 & 3/2 \end{pmatrix} \Rightarrow \begin{pmatrix} \vec{\mu_1} \\ \vec{\mu_2} \\ \vec{\mu_3} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3/2 & 1/2 \\ 1 & 1/2 & 3/2 \end{pmatrix}\begin{pmatrix} \vec{\alpha_1} \\ \vec{\alpha_2} \\ \vec{\alpha_3} \end{pmatrix}$$

$$\vec{\mu_1} = \frac{1}{2}(2\vec{\alpha_1} + \vec{\alpha_2} + \vec{\alpha_3}) = (1\ 0\ 0), \quad \vec{\mu_2} = \frac{1}{2}(\vec{\alpha_1} + 3/2\vec{\alpha_2} + 1/2\vec{\alpha_3}) = (1/2\ 1/2\ -1/$$

$$\vec{\mu_3} = \frac{1}{2}(\vec{\alpha_1} + \vec{\alpha_2} + 3\vec{\alpha_3}) = (1/2\ 1/2\ 1/2)$$

$$\vec{\mu}^* = \lambda_1\vec{\mu}_1 + \lambda_2\vec{\mu}_2 + \lambda_3\vec{\mu}_3 = \left(\lambda_1 + \frac{\lambda_2+\lambda_3}{2}, \frac{\lambda_2+\lambda_3}{2}, \frac{-\lambda_2+\lambda_3}{2}\right)$$

$$\dim(\vec{\mu}^*) = \prod_{\alpha\in\Delta^+}\left(1 + \frac{\vec{\mu}^*\cdot\alpha}{\rho\cdot\alpha}\right)$$

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha = (2, 1, 0)$$

$$\Rightarrow \boxed{\begin{aligned}\dim(\vec{\mu}^*) &= (1+\lambda_1)(1+\lambda_2)(1+\lambda_3)\\ &\cdot\left(1+\frac{\lambda_1+\lambda_2}{2}\right)\left(1+\frac{\lambda_1+\lambda_3}{2}\right)\cdot\\ &\left(1+\frac{\lambda_1+\lambda_2+\lambda_3}{3}\right)\end{aligned}}$$

$$\rho\cdot\alpha_1 = 1 \qquad \vec{\mu}^*\cdot\alpha_1 = \lambda_1$$

$$\rho\cdot\alpha_2 = 1 \qquad \vec{\mu}^*\cdot\alpha_2 = \lambda_2$$

$$\rho\cdot\alpha_3 = 1 \qquad \vec{\mu}^*\cdot\alpha_3 = \lambda_3$$

$$\rho\cdot\alpha_4 = 2 \qquad \vec{\mu}^*\cdot\alpha_4 = \lambda_1+\lambda_3$$

$$\rho\cdot\alpha_5 = 3 \qquad \vec{\mu}^*\cdot\alpha_5 = \lambda_1+\lambda_2+\lambda_3$$

$$\rho\cdot\alpha_6 = 2 \qquad \vec{\mu}^*\cdot\alpha_6 = \lambda_1+\lambda_2$$

$$[E_{\alpha_1}, E_{-\alpha_1}] = n_{1/2} - n_{-1/2} = Q - S_z = \vec{\alpha}_1\cdot\vec{H} \qquad \text{and } [E_{\alpha_{4,5,6}}, E_{-\alpha_{4,5,6}}]$$

$$[E_{\alpha_2}, E_{-\alpha_2}] = n_{-1/2} - n_{-3/2} = S_z - n_4 = \vec{\alpha}_2\cdot\vec{H} \qquad = \vec{\alpha}_{4,5,6}\cdot\vec{H}$$

$$[E_{\alpha_3}, E_{-\alpha_3}] = n_{3/2} - n_{1/2} = S_z + n_4 = \vec{\alpha}_3\cdot\vec{H}$$

Casimir $$C = \sum_{1\le a<b\le6} L_{ab}^2 = Q^2 + S_z^2 + n_4^2 + \sum_{\alpha\in\Delta^+}\{E_\alpha, E_{-\alpha}\}$$

$$= Q^2 + S_z^2 + n_4^2 + \sum_{\alpha\in\Delta^+}2E_{-\alpha}E_\alpha + [E_\alpha, E_{-\alpha}]$$

Acting on $|\mu^*\rangle = (m_Q, m_{S_z}, m_{n_4}) = \left(\lambda_1 + \frac{\lambda_2+\lambda_3}{2}, \frac{\lambda_2+\lambda_3}{2}, -\frac{\lambda_2+\lambda_3}{2}\right)$

$$\Rightarrow C = Q^2 + S_z^2 + n_4^2 + \sum_{\alpha\in\Delta^+}[E_\alpha, E_{-\alpha}] = Q^2 + S_z^2 + n_4^2 + \sum_{\alpha\in\Delta^+}\vec{\alpha}\cdot\vec{H}$$

$$C = Q^2 + S_z^2 + n_4^2 + 2\vec{p}\cdot\vec{H}$$

$$= Q^2 + S_z^2 + n_4^2 + 4Q + 2S_z$$

$$= m_Q(m_Q+4) + m_{S_z}(m_{S_z}+2) + m_{n_4}^2$$

Representations

λ_1 λ_2 λ_3	$\mu^*(m_Q, m_{S_z}, m_{n_4})$	$d(\mu^*)$	C
0 0 1	½ ½ ½	4	15/4
0 1 0	½ ½ -½	4	15/4
1 0 0	1 0 0	6	5
0 0 2	1 1 1	10	9
0 2 0	1 1 -1		
0 1 1	1 1 0	15	8
2 0 0	2 0 0	20	12

Application: spin-3/2 Hubbard model

$$H = -\wp + t \sum C_\alpha^\dagger(i) C_\alpha(j) - \mu \sum_\alpha n_\alpha(i)$$
$$\alpha = \pm 3/2, \pm 1/2$$

$$+ U_0 \sum P_0^\dagger(i) P_0(i) + U_2 \sum P_{2m}^\dagger(i) P_{2m}(i)$$

where $\quad P_0^\dagger(i) = \sum_{\alpha\beta} \langle 00 | \tfrac{3}{2}\alpha \tfrac{3}{2}\beta\rangle \, C_\alpha^\dagger C_\beta^\dagger(i)$

$$P_{2m}^\dagger(i) = \sum_{\alpha\beta} \langle 2m | \tfrac{3}{2}\alpha \tfrac{3}{2}\beta\rangle \, C_\alpha^\dagger C_\beta^\dagger(i) .$$

The hopping term has the SU(4) symmetry, i.e. the SO(6) symmetry, since all the 4 spin components are equivalent to each other.

How about the on-site term?   Check energy leve degeneracy

degeneracy

E:		degeneracy
0	—	1
$-\mu$	⤊ ⤊ ⤋ ⤋	4
$U_2 - 2\mu$	⤊⤊ ⤊⤋ $\frac{1}{\sqrt{2}}(⤊⤋ + ⤋⤊)$ ⤊⤋ ⤋⤋	5
$U_0 - 2\mu$	$\frac{1}{\sqrt{2}}(⤊⤋ - ⤋⤊)$	1
$\frac{U_0}{2} + \frac{5U_2}{2} - 3\mu$	⤊⤊⤋ ⤊⤊⤋ ⤊⤋⤋ ⤊⤋⤋	4
$U_0 + 5U_2 - 4\mu$	⤊⤊⤋⤋	1

quartet ⟷ Sp(4) spinor

quintet ⟷ Sp(5) vector

singlet ⟷ singlet

(Ex:) prove the energy of the 16 onsite states.

Proof: The energies of the empty, single particle, double-occupied states are obvious. Let's consider the 3-occupied state.

★ $|\psi\rangle = C^\dagger_{3/2} C^\dagger_{1/2} C^\dagger_{-1/2} |0\rangle$

$P^\dagger_0 = \frac{1}{\sqrt{2}} (C^\dagger_{3/2} C^\dagger_{-3/2} - C^\dagger_{1/2} C^\dagger_{-1/2})$ and $P_0 = \frac{1}{\sqrt{2}} (C_{-3/2} C_{3/2} - C_{-1/2} C_{1/2})$

$\Rightarrow$ $P_0 |\psi\rangle = \frac{-1}{\sqrt{2}} C^\dagger_{3/2} |0\rangle$ $\Rightarrow$ $\langle\psi| P^\dagger_0 P_0 |\psi\rangle = \frac{1}{2} \langle 0| C_{3/2} C^\dagger_{3/2} |0\rangle = \frac{1}{2}$

$\begin{cases} P^\dagger_{22} = C^\dagger_{3/2} C^\dagger_{1/2}, \quad P^\dagger_{21} = C^\dagger_{3/2} C^\dagger_{-1/2}, \quad P^\dagger_{20} = \frac{1}{\sqrt{2}} (C^\dagger_{3/2} C^\dagger_{-3/2} - C^\dagger_{1/2} C^\dagger_{-1/2}) \quad P^\dagger_{2-1} = C^\dagger_{-1/2} C^\dagger_{-3/2}, \\[2mm] P^\dagger_{2-2} = C^\dagger_{-1/2} C^\dagger_{-3/2} \end{cases}$

$\Rightarrow$ $P_{22} |\psi\rangle = C_{1/2} C_{3/2} |\psi\rangle = C^\dagger_{-1/2} |0\rangle$ $\Rightarrow$ $\langle\psi| \underset{P_{22}}{\underbrace{P^\dagger_{22}}} |\psi\rangle = 1$

Similarly: $\langle\psi| \underset{P^\dagger_{21}}{\underbrace{P_{21}}} |\psi\rangle = 1$

$\langle\psi| P^\dagger_{20} P_{20} |\psi\rangle = \frac{1}{2}$ and $\langle\psi| P^\dagger_{2-1} P_{2-1} |\psi\rangle = \langle\psi| P^\dagger_{2-2} P_{2-2} |\psi\rangle = 0$

hence $\langle\psi| H_u |\psi\rangle = \frac{u_0}{2} + \frac{5u_2}{2} - 3\mu$

★ For the fully occupied state $|\psi'\rangle = C^\dagger_{3/2} C^\dagger_{1/2} C^\dagger_{-1/2} C^\dagger_{-3/2} |0\rangle$

we have $P_0 |\psi'\rangle = \frac{1}{\sqrt{2}} [C^\dagger_{3/2} C^\dagger_{-3/2} - C^\dagger_{1/2} C^\dagger_{-1/2}) |0\rangle$

$\Rightarrow \langle\psi'| P^\dagger_0 P_0 |\psi'\rangle = 1$, similarly $\langle\psi'| P^\dagger_{2m} P_{2m} |\psi'\rangle = 1$

$\Rightarrow \langle\psi'| H_u |\psi'\rangle = u_0 + 5u_2 - 4\mu$

(※) Fiertz identity (three onsite SO(5) invariants)

$$\sum_{1\le a<b\le5} L_{ab}^2(i) + \sum_{1\le a\le5} n_a^2(i) + \frac{5}{4}\underbrace{(n-2)^2}_{(i)} = 5 \leftarrow \text{for each site}$$

In terms of SU(4), on each site we have one ~~column~~ column Rep.

	$L_{ab}^2 + n_a^2$	L_{ab}^2	n_a^2	$(n-2)^2$
n=0,2 empty / fully occupied	0	0	0	4
n=1,3 dim=4	15/4	5/2	$\frac{5}{4}$	1
n=2 quintet	5	4	1	0
singlet		0	5	

We can use two of them — Set

$$H_u = C_1\, n_a^2(i) + C_2 (n_i-2)^2 - (\mu-\mu_0) n_i + \text{const}$$

$$\Rightarrow n=2 \text{ sector} \Rightarrow C_1\cdot[5-1] = U_0 - U_2 \Rightarrow C_1 = \frac{U_0-U_2}{4}$$

at $\mu = \mu_0$, we have p-h symmetry $\Rightarrow$ $0 = U_0 + 5U_2 - 4\mu_0$

$$-\mu_0 = \frac{U_0+5U_2}{2} - 3\mu_0$$

$$\Rightarrow \mu_0 = \frac{U_0+5U_2}{4}$$

~~⇒ so check the n=0,4 state ⇒ C₂ 4~~

Let us set $\mu = \mu_0$, then the states with $n=0$ and 4, $E=0$

$$\Rightarrow -4C_2 = \text{const}$$

the states with $n = 1, 3,$ $\qquad E = -\mu_0 \quad \rule{1cm}{0.5pt}$

$\Rightarrow \quad C_1 \cdot \frac{5}{4} + C_2 + \quad (-4C_2) = -\mu_0 = .$

$$-3C_2 = -\mu_0 - \frac{5}{4}C_1 \Rightarrow C_2 = \frac{1}{3}\left[\mu_0 + \frac{5}{4}C_1\right] = \frac{3\mu_0 + 5\mu_2}{16}$$

$\Rightarrow$
$$H_u = \sum_i \frac{3\mu_0 + 5\mu_2}{16}\left[n(i) - 2\right]^2 + \frac{\mu_0 - \mu_2}{4} n_a^2(i)$$

$$- (\mu - \mu_0) \sum_i n(i)$$

explicit $SO(5)$, or $Sp(4)$ symmetry.

If $\mu_0 = \mu_2 \Rightarrow H_u$ only depends on n, hence

it's $SU(4)$ invariant.

$Sp(4)$ symmetry is free of fine-tuning. It's status in

spin $\frac{3}{2}$ systems is similar to $\cancel{0}$ $SU(2)$ in spin-$\frac{1}{2}$ systems.