4. Group series, Solvable groups

If a group is not a simple group, we need to decompose it into normal subgroup and the corresponding quotient group. If the normal subgroup is still not simple, this process can be kept on. This generates a series of normal subgroups.

Definition: $\boxed{\text{Normal series}}$ is a finite series of subgroups of G: $G = G_0 \triangleright G_1 \triangleright \cdots\ G_s = \{e\}$. ("$\triangleright$" means the group on the RHS is a normal subgroup on the LHS). G_{i-1}/G_i is called the factor of the series, i.e. a quotient group, $i=1, 2, \cdots s$. The length of the the normal series is "s", i.e. the # of factors.

Example: ① $S_4 \triangleright A_4 \triangleright B_4 \triangleright \{(1)\}$ is a length 3 normal series of S_4.

Comment: in the point group language, S_4 is T_d (the full symmetry group of a tetrahedral with 24 elements), A_4 is T (the proper sym operations of a tetrahedral), B_4 is the Klein group (D_2).

② $S_4 \triangleright B_4 \triangleright \{(1)\}$ is a length 2 normal series.

S_4:

$$S_4: \left\{ \begin{array}{l} \left.\begin{array}{l} \underbrace{(1),\ \{(12)(34),\ (13)(24),\ (14)(23)\}}_{B_4}, \\ \{(123),\ (132),\ (124),\ (142),\ (134),\ (143),\ (234),\ (243)\} \end{array}\right\} A_4 \\ \{(12),\ (34),\ (13),\ (24),\ (14),\ (23)\} \\ \{(1234),\ (1342),\ (1423),\ (1324),\ (1432),\ (1243)\} \end{array} \right.$$

In both series, the factor groups are not always simple.

For example: $B_4 \triangleright \{(1)\}$, $\rightarrow B_4/\{(1)\} = B_4$, and B_4 is not simple.

$S_4 \triangleright B_4 \rightarrow S_4/B_4 = S_3$ and S_3 is not simple.

This is like that we do not completely factorize, and we should further refine it, until every factor group is already simple. Then we arrive the composite series.

Note: normal subgroups do not have transitivity, for example,

$A_4 \triangleright B_4 \triangleright C_2$, but C_2 is NOT a normal subgroup of A_4.

Then we have: $\boxed{\underbrace{S_4 \triangleright A_4}_{C_2} \underbrace{\triangleright B_4}_{C_3} \underbrace{\triangleright C_2}_{C_2} \underbrace{\triangleright \{(1)\}}_{C_2} \quad \leftarrow \text{factor groups}}$

The composite series of a finite group may not be unique. We have the following Jordan - Holder theorem:

$\boxed{\text{Any two composite series of a finite group are isomorphic.}}$

example: ① $\underbrace{C_6 \triangleright C_3}_{C_2} \underbrace{\triangleright \{(1)\}}_{C_3}$

② $\underbrace{C_6 \triangleright C_2}_{C_3} \underbrace{\triangleright \{(1)\}}_{C_2}$

Both are composite series of C_6, Their sets of factor groups are the same up to a permutation.

☆ $\boxed{\text{Solvable groups}}$ ← a generalization of Abelian group

Definition: If a normal series of a group G,

$$G = G_0 \rhd G_1 \rhd \cdots \quad G_s = \{(e)\},$$ satisfies that

all G_{i-1}/G_i's are Ablieaen groups, then G is called a solvable group, and this series is called a solvable series.

- All Abeliean groups are solvable
- S_3 and S_4 are solvable : $S_3 \rhd C_3 \rhd \{e\}$,
  $S_4 \rhd A_4 \rhd B_4 \rhd \{e\}$.
- Non-abeliean simple groups are not solvable.

For example: A_5 : icosahedron group

A_5:

$\{e\}$; ⓵ 1

$\{(12)(34), (12)(45), (12)(35); (13)(24), (13)(25), (13)(45);$
$(14)(23), (14)(25), (14)(35); (15)(23), (15)(24), (15)(34);$
$(23)(45), (24)(35), (25)(34)\}$; $\binom{5}{2}\binom{3}{2}/2 = 15$

$\{(123), (132); (124), (142); (125), (152); (134), (143); (135)(153);$
$(145), (154); (234), (243); (235), (253); (245), (254); (345), (354)\}$;
$\binom{5}{3}\times 2 = 20$

$\{(12345), (1\#\#\#\#)\}$ ← permutation of 2345 $\quad 4! = 24$.

60 - elements

A_5 does not have non-trivial normal subgroups! Ironically, A_5 is simple!

- When $n \geq 5$, S_n is not solvable.
- The factor groups of the normal series of a solvable group are all solvable.

- **Theorem**: G is a solvable finite group. Then G has a composite series such that for every factor group, its order is a prime number.

Proof: Since G is solvable, it has a solvable normal series

$G = G_0 \triangleright G_1 \triangleright \cdots \quad G_s = \{e\}$, among which G_{i-1}/G is an Abelian group. If its order is a composite number n, then it has a subgroup H, whose order is a factor of G_{i-1}/G. This can be shown as follows: pick up an element $a \in G_{i-1}/G$, and consider the cyclic group generated by $\{a^k\}$, $k = 1, \cdots m$. If $m < n$, then $\{a^k\}$ is a subgroup of G_{i-1}/G, and m is a factor of n. If $m = n$, assume $n = n_1 n_2$, then switch the generator to a^{n_1}, and check the cyclic group $\{(a^{n_1})^k\}$, $k = 1, \cdots n_2 < n$. Since groups are Abelian, H is a normal subgroup. Then we can add H, into

$$G_{i-1} \triangleright H \triangleright G_i,$$

and keep on repeating the process, until each factor group is at the prime order. i.e

$$\boxed{\underbrace{G_0 \triangleright G_1}_{G_0/G_1 = C_{p_1}} \underbrace{\triangleright G_2}_{C_{p_2}} \cdots \cdots \underbrace{G_s \triangleright \{e\}}_{C_{p_s}}}$$

About the solvability of p-groups (p-prime number)

- A p-group is that ~~its every element~~ the order of its every element is a power of P, i.e.

$\{a^k\}$. $k = 0, 1, 2, \cdots p^n - 1$, is a p^n-order cyclic group.

Different elements can have different orders.

- p-groups can be different at the same order.

  D_2 and C_4 are order 4, — Ablican p-groups are called prima

  D_4 and Q (quaternion group) are order 8. (non-abelian)

- A finite group is p-group iff its order is p^n.

  $n=1$, p-order cyclic group;

  $n=2$. $C_p \otimes C_p$, its also Abelian.

- A finite p-group has a nontrivial center Z.

- $\boxed{\text{All finite p-groups are solvable.}}$

  proof by induction: ① $n=1$, p-th order cyclic group it's of course solvable

② Assume $p^1, \cdots p^{n-1}$, order groups are solvable. Then Consider p^n-group G, its center is denoted as Z. If Z is the same as p^n-group, then its already Abelian, hence, solvable. If not, assum Z's order p^k $(1 \leq k \leq n-1)$, which is solvable, then G/Z is the order of p^{n-k}, which is solvable. $\Rightarrow$ G is solvable.

Appedix: Theorem: the alternating groups A_n are simples when $n \geq 5$

Proof: A_n is the normal subgroup of S_n, only containing even permutatio

If we assume a normal subgroup of A_n denoted as H, which is not $\{(1)\}$. We prove that it must be A_n itself.

We express every permutation in H as products of rotations without overlapping elements, say, $(12)(34)\cdots$. Among them, we pick up the one which changes the smallest amount of numbers, and denote it as "h". Then "h" satisfies the following properties

i) The lengths of rotations in h must be the same. Otherwise, we set $h = (\alpha_1 \alpha_2 \cdots \alpha_k)(\beta_1 \cdots \beta_k \beta_{k+1} \cdots)(\cdots)$, then $h^k = (\beta_1 \beta_{k+1} \cdots)(\cdots)$. The rotatio of $(\alpha_1 \cdots \alpha_k)$ disappears, but other rotations keep the same lengths. Hence, h^k's length is reduced!

ii) The length of each rotation cannot be longer than 4. Otherwise, Set $h = (\alpha_1 \alpha_2 \alpha_3 \alpha_4 \gamma_1 \cdots \gamma_l)(\cdots)$. Consider an even permutation $g = (\alpha_2 \alpha_3 \alpha_4)$ in A_n, and calculate $h_1 = g\, h\, g^{-1} = ?$

$$h_1 = \begin{pmatrix} \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_3 & \alpha_4 & \alpha_2 \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \gamma_1 & \cdots & \gamma_l \\ \alpha_2 & \alpha_3 & \alpha_4 & \gamma_1 & \gamma_2 & \cdots \gamma_l & \alpha_1 \end{pmatrix} \begin{pmatrix} \alpha_3 & \alpha_4 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_3 & \alpha_4 & \alpha_2 & \gamma_1 & \cdots & \gamma_l \\ \alpha_3 & \alpha_4 & \alpha_2 & \gamma_1 & \gamma_2 & \cdots & \alpha_1 \end{pmatrix}(\cdots)$$

Then calculate $h_1 h^{-1} = ?$

$$h^{-1} = \begin{pmatrix} \alpha_2 & \alpha_3 & \alpha_4 & \gamma_1 & \cdots & \gamma_l & \alpha_1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \gamma_1 & \cdots & \gamma_l \end{pmatrix} \Rightarrow$$

$$h_1 h^{-1} = \begin{pmatrix} \alpha_1 & \boxed{\alpha_2 \ \alpha_3 \ \gamma_1} & \gamma_2 & \cdots & \gamma_l & \alpha_4 \\ \alpha_1 & \boxed{\alpha_3 \ \gamma_1 \ \alpha_2} & \gamma_2 & & \gamma_l & \alpha_4 \end{pmatrix} = (\alpha_2 \alpha_3 \gamma_1) \leftarrow \text{length-3 rotation!}$$

iii) h can only be a single rotation rather than a product of rotations

(a) if h is a product of two length-two rotations, say,

$h=(a_1 a_2)(a_3 a_4)$. We take $g=(a_1 a_2 a_5)\in A_n$, then

$$g^{-1}hg = \begin{pmatrix} a_2 & a_5 & a_1 \\ a_1 & a_2 & a_5 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}\begin{pmatrix} a_3 & a_4 \\ a_4 & a_3 \end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_5 \\ a_2 & a_5 & a_1 \end{pmatrix} = \begin{pmatrix} a_1 & a_5 \\ a_5 & a_1 \end{pmatrix}\begin{pmatrix} a_3 & a_4 \\ a_4 & a_3 \end{pmatrix}$$

$$h(\underbrace{g^{-1}hg}_{\in H}) = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}\begin{pmatrix} a_3 & a_4 \\ a_4 & a_3 \end{pmatrix}\begin{pmatrix} a_1 & a_5 \\ a_5 & a_1 \end{pmatrix}\begin{pmatrix} a_3 & a_4 \\ a_4 & a_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}\begin{pmatrix} a_1 & a_5 \\ a_5 & a_1 \end{pmatrix} = \begin{pmatrix} a_1 & a_5 & a_2 \\ a_5 & a_2 & a_1 \end{pmatrix}$$

hence $(a_1 a_5 a_2)\in H$, and it changes three objects less than h. This is a contradiction.

(b) if h is a product of two length-three rotation.

$h=(a_1 a_2 a_3)(a_4 a_5 a_6)$, we take $g=(a_1 a_2 a_5)$, then

$$h^{-1}(g^{-1}hg) = \begin{pmatrix} a_2 & a_3 & a_1 \\ a_1 & a_2 & a_3 \end{pmatrix}\begin{pmatrix} a_5 & a_6 & a_4 \\ a_4 & a_5 & a_6 \end{pmatrix}\begin{pmatrix} a_2 & a_5 & a_1 \\ a_1 & a_2 & a_5 \end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_1 \end{pmatrix}\begin{pmatrix} a_4 & a_5 & a_6 \\ a_5 & a_6 & a_4 \end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_5 \\ a_2 & a_5 & a_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 & a_5 & a_3 & a_4 \\ a_2 & a_5 & a_3 & a_4 & a_1 \end{pmatrix}$$ — length-5 rotation, less objects involved!

$\Rightarrow$ in Summary, "h" can only be a single rotation of lengths two or three. Since h is an even permutation, hence, h is a rotation of length-3. Since H is a normal subgroup, hence

$\boxed{\text{H contains all length-3 rotations.}}$

~~Since~~ Every pertumulation can be represented as products of exchanges, and even permutation are expressed as even #'s of exchanges: $g=t_1t_2\cdots t_{2k-1}t_{2k}$

① if $t_1 = t_2 \Rightarrow t_1 t_2 = \{(1)\}$

② if t_1, t_2 share one #, $t_1 t_2 = (\alpha\beta)(\alpha\gamma) = (\alpha\gamma\beta)$

③ if t_1, t_2 have no commen #'s, $t_1 t_2 = (\alpha\beta)(\gamma\delta) = (\alpha\gamma\beta)(\gamma\delta\alpha)$

Hence, all the length-3 rotations generate An!