Lect 4: Orthogonality, Characters

★ **Equivalent representation**: If for two representations $M'(g)$ and $M(g)$, they can be related by $M'(g) = C^{-1} M(g) C$, $\forall g$, then we say $M'(g)$ and $M(g)$ are equivalent. They can be related by a change of basis. We are ~~interested~~ in finding out non-equivalent representations.

★ **Reducibility and irreducibility**

Consider a vecter space V. If there exists a subspace $\mathcal{U} \subseteq V$, such that for all $M(g)$, and a vecter $x \in \mathcal{U}$, they satisfy

$$\boxed{M(g)\, x \in \mathcal{U}, \quad \forall g \in G, \text{ and } x \in \mathcal{U}.}$$

then $\mathcal{U}$ is an invaraint subspace, and $M(g)$ is reducible.

Then the matrices $M(g)$ take the form

$$\boxed{M(g) = \begin{array}{c} \left[\begin{array}{c|c} M_1 & N \\ \hline 0 & M_2 \end{array}\right] \end{array} \begin{array}{l} \}\, \mathcal{U} \\ \}\, \mathcal{U}' \end{array}}$$

If $M(g)$ is an unitary matrix for all $g \in G$, the $M(g)$ is called unitary representation, the N is also 0. (please do as an excise)

For a unitary representation, $M(g)$ can be represented as.

$$\boxed{M(g) = \begin{bmatrix} M_1(g) & 0 \\ 0 & M_2(g) \end{bmatrix} \begin{matrix} \}\, \mathcal{U} \\ \}\, \mathcal{U}' \end{matrix} \quad \text{for } \forall g \in G}$$

Completely reducible, $\mathcal{U}' \oplus \mathcal{U} = V$, where $\oplus$ means direct sum. $\mathcal{U}$ and $\mathcal{U}'$ are orthogonal complement to each other.

Certainly, V can be further decomposed into $V = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \cdots \oplus \mathcal{U}_m$ and $M(g)$ is block-diagonalized in each subspace. If each subspace cannot be further decomposed, then the representation in each subspace is called irreducible.

~~We~~ Our goal is to identify all the non-equivalent irreducible representations of a group G.

Schur's lemma

① Assume that $M^{(1)}(G)$ and $M^{(2)}(G)$ are two non-equivalent irreducible representations of G, whose dimensions are m_1 and m_2 respectively. X is an $m_1 \times m_2$ dimensional matrix, If for any $g \in G$, we have

$$M^{(1)}(g)\, X = X\, M^{(2)}(g), \text{ then } X = 0.$$

Proof: $M^{(1)}(g)$ is an $m_1 \times m_1$ matrix, $M^{(2)}(g)$ is an $m_2 \times m_2$ matrix and X is an $m_1 \times m_2$ matrix.

① $m_1 > m_2$: We view X as a collection of m_2 column vector

$$X = [y_1, \cdots y_{m_2}], \text{ and } y_i = \begin{pmatrix} y_{1i} \\ \vdots \\ y_{m_1 i} \end{pmatrix}.$$

Then $M^{(1)}(g)[y_1, y_2 \cdots y_{m_2}] = [y_1, y_2 \cdots y_{m_2}]\, M^{(2)}(g)$

$\Rightarrow M^{(1)} y_i = \sum_{j=1}^{m_2} y_j\, M^{(2)}_{ji}(g)$, which means that there exists an invariant space with dimensions $< m_1$ for $M^{(1)}(G)$. Since $M^{(1)}(G)$ is irreducible, it has to zero, i.e $X = 0$.

② $m_1 = m_2$: If $\det X \neq 0$, $\Rightarrow M^{1}(g) = X M^{(2)}(g) X^{-1}$

$\Rightarrow M^{(1)}(G)$ is equivalent to $M^{(2)}(G)$, which is contridict to the assumption. If $\det X = 0$, which means $X = \{y_1, \cdots y_{m_1}\}$, then the number of linearly indepentent vectors $< m_1$. Hence for the space spanned by $y_1, \cdots y_{m_1}$; its dimensions $< m_1$.

According to ① $\Rightarrow$ X has to be zero.

③ $m_1 < m_2$, we take transpose $\Rightarrow M^{(2)T}(g)\, X^T = X^T M^{(1)T}(g)$

then X^T is an $m_2 \times m_1$ matrix with $m_2 > m_1$. If $M^{(2)(T)}(G)$ is non-reducible, i.e. has no $\overbrace{\text{invaiant}}^{}$ non-zero subspace, then we can apply ①, to arrive $X^T = 0$.

If $M^{(2),T}(G)$ is reducible, then there exists a square matrix Y

$$Y^{-1} M^{(2),T}(G)\, Y = \begin{array}{c} \left[\begin{array}{c|c} M' & N \\ \hline 0 & M'' \end{array}\right] \begin{array}{l} \}\, l \\ \}\, m_2 - l \end{array} \end{array}$$

Then

$$Y^T M^{(2)}(G) (Y^{-T})^{-1} = \left[\begin{array}{c|c} M'^T & 0 \\ \hline N^T & M''^T \end{array}\right].$$

Then for states in the subspace $\begin{array}{c} \left[\begin{array}{c} 0 \\ \hline \omega \end{array}\right] \begin{array}{l} \}\, l \\ \}\, m_2 - l \end{array} \end{array}$,

they form an invariant space, which means $M^{(2)}(G)$ is reducible, and thus is contradicted to the assumption.

② Consider a irreducible representation $M(G)$ for a group G. If X commutes with all $M(g)$ for $\forall g \in G$, then $X = \lambda I$, where λ is a constant.

Proof: Any finite dimensional square matrix at least has one eigenvalue and one eigenvector. We assume the eigenvalue is λ, and define $Y = X - \lambda I$.

Then $M(g) Y = Y M(g)$ with $\det Y = 0$. According the the case ② in ①, we have $Y = 0$, i.e $X = \lambda I$.

§ Characters and orthogonality

(*) unitary representation:

Theorem 1: Any linear representation of a finite group is equivalent to an unitary representation. Any two equivalent unitary Reps can be related through unitary transformations.

Proof: For a given Rep. $M(G)$, we want find a similar transformation X such that $\bar{M}(g) = X^{-1} M(g) X$, such that $\bar{M}^\dagger(g)\bar{M}(g) = 1$, $\forall g \in G$.

Let's figure out what kind X can do the job: $X^\dagger M^\dagger(g)(X^\dagger)^{-1} X^{-1} M(g) X = 1$

or, $\boxed{M^\dagger(g)(XX^\dagger)^{-1} M(g) = (XX^\dagger)^{-1}}$ ← what kind of $XX^\dagger$ can lead to this?

define $(XX^\dagger)^{-1} = \sum_{g' \in G} M^\dagger(g') M(g')$, then check

then
$$M^\dagger(g)\Big(\sum_{g'\in G} M^\dagger(g') M(g')\Big) M(g) = \sum_{g'\in G} [M(g')M(g)]^\dagger M(g')M(g)$$
$$= \sum_{g'\in G} M^\dagger(g'g)\, M(g'g) = \sum_{g} M^\dagger(g) M(g) = (XX^\dagger)^{-1}.$$

For finite group, for a fix g, $g'g$ with g' runing ~~across~~ over in G, the $g'g$ also covers every element in G.

Then the question remaining is: Can we find X, satisfying

$$\boxed{(XX^\dagger)^{-1} = \sum_{g\in G} M^\dagger(g) M(g)}$$

It's easy to show that $\sum M^\dagger(g)M(g)$ is Hermitian. It can also be proven that it's positive-definite. Hence it can diagonalized with positive eigenvalues

$$\sum_g M^\dagger(g)M(g) = u^\dagger \Gamma u, \quad \text{where } \Gamma = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

then define $\Gamma' = \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix}$, $\Rightarrow u^\dagger \Gamma u = u^\dagger \Gamma' u \; u^\dagger \Gamma' u$

hence we ~~do~~ define $X^{-1} = u^\dagger \Gamma' u$, then $(X^{-1})^\dagger = X^{-1}$

or $\boxed{X = u^\dagger \begin{bmatrix} \lambda_1^{-1/2} & & \\ & \vdots & \\ & & \lambda_n^{-1/2} \end{bmatrix} u}$, ← satisfy the requirement.

- The proof of the 2nd part is left as an exercise.

Comment: The proof relies on the convergence of $\sum_g M^\dagger(g)M(g)$, which is obervious for a finite group. It may be ok for compact Lie group, for which we need to define integral over group manifold. But for non-compact Lie group, this theorem doesnot work any more. For example, for the Lorentz group $SO(3,1)$, the spinor Reps are not unitary.

④ orthogonality relation

Theorem: Consider two nonequivalent irreducible unitary Reps $M(G)$ and $N(G)$ of a finite group G. Then as a vector in the group space, $M(G)$ and $N(G)$ satisfy the orthogonal relation

$$\boxed{\sum_{g\in G} M^*_{\mu\rho}(g)\, N_{\nu\lambda}(g) = \frac{h}{m}\, \delta_{MN}\, \delta_{\mu\nu}\, \delta_{\rho\lambda}},$$

where h is the order of G, m is dimension of the representation of M, or N. δ_{MN} means that M and N needs to be the same Rep.

Preparation: we define the group space. For a finite group, we take each of its element as a basis, and define a linear space. Such a space is called the group space. It's dimension is just the order of the group. A vector in the group space can be represented as

$$X = \sum_{g\in G} C_g\, g .$$

In an ordinary linear space, only the sum of two vectors is defined. If we further define product between vectors, such that it's closed for multiplication; $X \in L$, $Y \in L$, then $XY \in L$ and $Z(X+Y) = ZX + ZY$, then such a linear space is called an algebra.

The group algebra is defined as:

$$xy = \left(\sum_{g\in G} c_g\, g\right)\left(\sum_{g'\in G} d_{g'}\, g'\right) = \sum_{g,g'} c_g d_{g'} (gg')$$

$$= \sum_t \left\{ \sum_s c_{ts^{-1}} d_s \right\} t.$$

Now Let's prove the theorem: Assume the dimensions of the Reps M and N as m and n, respectively. Define an $m\times n$ matrix $Y(\mu\nu)$ as

$\boxed{Y(\mu\nu)_{\rho\lambda} = \delta_{\mu\rho}\delta_{\nu\lambda}}$, i.e, only its $\underbrace{\mu\text{th row and }\nu\text{th column}}_{\text{element at}}$ is nonzero. Then define an $m\times n$ matrix $X(\mu\nu)$ as

$$\boxed{X(\mu\nu) = \sum_{g\in G} M(g^{-1})\, Y(\mu\nu)\, N(g),}$$

Then $X(\mu\nu)_{\rho\lambda} = \sum_{g\in G}\sum_{\tau\tau'} M_{\rho\tau}(g^{-1})\, \delta_{\mu\tau}\delta_{\nu\tau'}\, N_{\tau'\lambda}(g) = \sum_{g\in G} M_{\rho\mu}(g^{-1})\, N_{\nu\lambda}(g)$

On the other hand, $X(\mu\nu)$ satisfies:

$$M(g)\, X(\mu\nu) = \sum_{g'\in G} M(g)\, M(g'^{-1})\, Y(\mu\nu)\, N(g')$$

$$= \sum_{g'} M(gg'^{-1})\, Y(\mu\nu)\, N(g') \qquad \text{set } f^{-1} = gg'^{-1} \Rightarrow g' = fg$$

$$= \left[\sum_f M(f)\, Y(\mu\nu)\, N(f)\right] N(g) = X(\mu\nu)\, N(g).$$

Hence, According Schur's lemma, if $M \neq N$, the $X(\mu\nu) \equiv 0$, i.e

$$X(\mu\nu)_{\rho\lambda} = 0 \Rightarrow \boxed{\begin{aligned} \sum_{g\in G} M_{\rho\mu}(g^{-1})\, N_{\nu\lambda}(g) &= \sum M^{\dagger}_{\rho\mu}(g)\, N_{\nu\lambda}(g) \\ &= \sum M^{*}_{\mu\rho}(g)\, N_{\nu\lambda}(g) = 0 \end{aligned}}$$

② If $M = N$, then Schul's lemma $\Rightarrow$ $X(\mu\nu)$ need to a constant matrix, i.e. $\underset{\rho\lambda}{X(\mu\nu)} = C(\mu\nu)\,\delta_{\rho\lambda}$, where $C(\mu\nu)$ is a $\mu\nu$-dependent const.

plug in
$$X_{\rho\lambda}(\mu\nu) = \sum_{g\in G} M^*_{\mu\rho}(g)\, M_{\nu\lambda}(g) = C(\mu\nu)\,\delta_{\rho\lambda}$$

set $\rho = \lambda$, sum over ρ $\Rightarrow$
$$m\, C(\mu\nu) = \sum_{g\in G} \sum_{\rho} M_{\rho\mu}(g^{-1})\, M_{\nu\rho}(g)$$
$$= \sum_{g\in G} \underset{\nu\mu}{M(gg^{-1})} = h\,\delta_{\mu\nu}$$

$$\Rightarrow \boxed{\sum_{g\in G} M^*_{\mu\rho}(g)\, M_{\nu\lambda}(g) = \frac{h}{m}\,\delta_{\mu\nu}\,\delta_{\rho\lambda}}$$

comments ① $M_{\alpha\beta}(g)$ can be viewed as a group function. Each representation contributes m^2 - orthogonal group functions. There are only h - basis, hence $\boxed{\sum_j m_j^2 \le h = |G|}$

⊛ Definition of Character:

Consider a Matrix Rep $M(G)$ for a group G. For each element $g \in G$, we define its character $\chi(g) = tr[M(g)]$ in the Rep $M(G)$.

comment: For all elements in a class C, we have

$$\chi(g' g g'^{-1}) = tr[M(g') M(g) M(g'^{-1})] = tr[M(g)] = \chi(g).$$

Hence. $\chi(g)$ is a function defined on class.

Corollary: The character $\chi_M(g)$ as a vector defined on the group space, they satisfy the orthogonal relation.

we use $$\sum_{g\in G} M^*_{\mu\rho}(g)\, N_{\nu\lambda}(g) = \frac{|G|}{\dim_M}\, \delta_{MN}\, \delta_{\mu\nu}\, \delta_{\rho\lambda}$$

set $\mu=\rho$, $\nu=\lambda$, and sum over μ, and ν, $\Rightarrow$

$$\sum_{g\in G} \chi^*_M(g)\, \chi_N(g) = \frac{|G|}{\dim M}\, \delta_{MN} \sum_{\mu\nu} \delta_{\mu\nu}\, \delta_{\mu\nu} = \frac{|G|}{\dim M}\, \delta_{MN} \dim M$$

$$= |G|\, \delta_{MN}$$

Since $\chi_M(g)$ is a class function, we can further simplify. Assume that G has n_c classes, and each class C_α contains $n(C_\alpha)$ elements

then $$\boxed{\sum_{\alpha=1}^{n_c} \left[\frac{n(C_\alpha)}{|G|}\right]^{1/2} \chi^*_M(C_\alpha) \left[\frac{n(C_\alpha)}{|G|}\right]^{1/2} \chi_N(C_\alpha) = \delta_{MN}}$$

Comments: ① $\chi_M(C_\alpha)$ can be viewed as a class function, or, a vector defined in $n(C_\alpha)$ dimensional vector space. Different Reps give rise to orthogonal vectors, hence. <u>the # of irreducible representat</u> <u>$\leq$ # of classes</u>, i.e. (dim of linear space).

② For two irreducible representations, the necessary and ~~sufficient~~ sufficient condition for them to be equivalent is that their characters $\chi(g)$ are the same for each $g \in G$.

we can use the character function $\chi_M(g)$ to decampse a reducible representation χ_D. If $D(G)$ can be decomposed into a sum of irreducible representation $\boxed{X^{-1} D(g) X = \oplus\, a_M\, M(g),}$ where a_M is an integer showing the times of respresentation $M(G)$. Then

$$\boxed{\chi_D = \sum_{M'} a_{m'} \chi_{M'}(g) \Rightarrow a_m = \frac{1}{|G|} \sum_{g\in G} \chi_M^*(g) \chi_D(g)}$$

and $\sum_g |\chi_D(g)|^2 = g \sum_m a_m^2 \geq g$, hence

③ A representation $D(G)$ for a finite group to be irreducible,
The necessary and sufficient condition for
is that $\boxed{\sum_g |\chi(g)|^2 = |G|}$

④ The completeness of representations $M^j(G)$

we have already had that for all the non-equivalent irreducible Reps,
we have

$$\begin{cases} \sum_j m_j^2 \leq |G| \\ \sum_j 1 \leq n_c \end{cases}.$$

We will prove that actually the inegualies are actually egualies.

Consider a special representation: The canonical representation, which uses the group space as the linear space of representation. To be concrete the matrix for $g \in G$, is defined as

M^c is a matrix.

$$\boxed{g \cdot k = \sum_{p \in G} p \, M^c_{pk}(g)} \quad \text{for } \forall k \in G.$$

g here works as an operation

k is also a group element, Here it works as a basis.

Then we can prove that $M^c(g)$ defines a representation.

$$g_2 g_1 \cdot k = \sum_{p \in G} p \, M^c_{pk}(g_2 g_1)$$

$$g_2 g_1 \cdot k = g_2 \sum_{p' \in G} p' \, M^c_{p'k}(g_1) = \sum_{p'} \sum_{p''} p'' \, M^c_{p''p'}(g_2) \, M^c_{p'k}(g_1)$$

$$= \sum_{p''} p'' \sum_{p'} M^c_{p''p'}(g_2) \, M^c_{p'k}(g_1) \quad \Rightarrow \boxed{M(g_2 g_1) = M(g_2) M(g_1)}$$

The concrete matrix $M^c(g)$ can be constructed

$$\boxed{M^c_{pk}(g) = \begin{cases} 1 & \text{if } p = kg \\ 0 & p \neq kg \end{cases}}$$

hence there's only one nonzero element in each row in $M_{pk}(g)$.

If $g = e \Rightarrow M^c_{pk}(e) = \delta_{pk} \Rightarrow \chi_{M^c}(e) = |G|$

$g \neq e \Rightarrow$ all diagonal terms of $M^c(g) = 0 \Rightarrow \chi_{M^c}(g) = 0$.

Now, we decompose M^c into irreducible reps M_j

$$\boxed{\begin{aligned} a_{M_j} &= \frac{1}{|G|}\sum_{g\in G}\chi^*_{M_j}(g)\,\chi_{M^c}(g) = \frac{1}{|G|}\chi^*_{M_j}(e)\,\chi_{M_c}(e) \\ &= \frac{1}{|G|}\cdot m_j\cdot|G| = m_j \end{aligned}}$$

Hence $X^{-1} M^c(g) X = \oplus\, m_j\, M_j(g)$

take the cheracter of the identity "e", $\Rightarrow$ $\boxed{|G| = \chi(e) = \sum_j m_j^2.}$

Then ① For All the non-equivalent irreducible representations $M^j(G)$, their elements $M^j_{\mu\nu}(g)$ as a vecter in the group space form a set of complete ~~orthogonal~~ basis. [Compare the # of basis]

Any group function $F(g)$ can be expanded as

$$\boxed{F(g) = \sum_{j,\mu\nu} C^j_{\mu\nu}\, M^j_{\mu\nu}(g) \Rightarrow C^j_{\mu\nu} = \frac{m_j}{|G|}\sum_g M^{j*}_{\mu\nu}(g)\,F(g).}$$

② Class function: if $F(g)$ takes the same value for all elements in a class, then $F(g)$ is a class function.

$$F(g) = \frac{1}{|G|}\sum_{s\in G} F(sgs^{-1}) = \sum_{j,\mu\nu}\frac{C^j_{\mu\nu}}{|G|}\sum_{s\in G} M^j_{\mu\rho}(s)\, M^j_{\rho\lambda}(g)\, M^j_{\lambda\nu}(s^{-1})$$

$$= \sum_{j,\mu\nu}\frac{C^j_{\mu\nu}}{|G|}\left\{\sum_s M^j_{\mu\rho}(s)\, M^{j*}_{\nu\lambda}(s)\right\} M^j_{\rho\lambda}(g)$$

$$\Rightarrow F(g) = \sum_{j,\mu\nu} \frac{C^j_{\mu\nu}}{|G|} \frac{|G|}{m_j} \sum_{\rho\lambda} \delta_{\mu\nu}\delta_{\rho\lambda} M^j_{\rho\lambda}(g)$$

$$= \sum_j \left[\frac{1}{m_j} \sum_\mu C^j_{\mu\mu}\right] \chi_{M_j}(g)$$

Hence, any class function can be represented by characters, i.e. the characters $\chi_{M_j}(g)$ of all the irreducible representations form a complete basis of class function space. More precisely

$$\boxed{\begin{aligned} F(g) &= F(sgs^{-1}) = \sum_j C_j \chi_{M_j}(g) \\ C_j &= \frac{1}{|G|}\sum_{g\in G} \chi^*_{M_j}(g) F(g) \end{aligned}}$$

Since $\chi_{M_j}(g)$ is complete, the # of irreducible representations = the dimension of class function space, i.e, # of classes.

In summary:

$$\boxed{\sum_j 1 = n_c, \quad \sum_j m_j^2 = |G| \qquad \sum_M \chi^*_M(C_\alpha)\chi_M(C_\beta) = \frac{|G|}{n_{C_\alpha}}\delta_{\alpha\beta} \\ \sum_g \chi^*_M(g)\chi_N(g) = |G|\delta_{MN},}$$

we also have
$$\sum_M \left(\frac{n_{C_\alpha}}{|G|}\right)^{1/2} \chi^*_M(C_\alpha) \left(\frac{n_{C_\beta}}{|G|}\right)^{1/2} \chi_M(C_\beta) = \delta_{\alpha\beta}$$

(prove it!).

The orthogonality - completeness of $\chi_M(g)$ in the space of class function means, we can preprare a table of characters as

	e	class α	class β	$\cdots$
χ_{M_1}	1	1	1	
χ_{M_2}		$\chi_{M_j}(C_\alpha)$		
$\vdots$				

→ The table can be viewed as a square matrix.

$$\sum_\alpha n_{C_\alpha} \chi^*_M(C_\alpha)\chi_N(C_\beta) = |G|\delta_{MN}$$

$$\sum_M \chi^*_M(C_\alpha)\chi_M(C_\beta) = \frac{|G|\delta_{\alpha\beta}}{|n_{C_\alpha}|}$$

Example: Abelian groups:

Every elements is a class $\Rightarrow \sum_{j=1}^{|G|} m_j^2 = |G| \Rightarrow m_j = 1.$

Hence, all representations are one dimensional!

	e	σ
A_1	1	1
A_2	1	-1

Z_2

m	e	R	R^2	$\cdots$	R^{N-1}
$m=0$	1	1	1	1	1
1	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	1	ω^m	ω^{2m}	$\cdots$	$\omega^{m(N-1)}$
$N-1$					

$\omega = e^{-i\frac{2\pi}{N}}$

C_N or Z_N group.

① The product of characters is also a character.

$$\chi_{M_1}(g)\chi_{M_2}(g) = \chi_M(g), \quad \forall g \in G$$

② The characters also form a group, which is isomorphically to the Abelian group G itself

(*) Real and pseudo-real representations

Consider a representation $M(g)$, then it's easy to see $M^*(g)$ also form a representation. If $M^*(g)$ is equivalent to $M(g)$, then we say, $M(g)$ is self-conjugate. Rep. If we can find a basis, in which $M(g)$ only have real entries, we call $M(g)$ as a real representation, otherwise, $M(g)$ is called pseodo-real.

Theorem: For a self-conjugate unitary representation of a finite group, the similar transformation between it and it complex conjugate Rep can only be a symmetric or antisymmetric matrix. For real representations, it is a symmetric matrix, while it is an antisymmetric matrix for the pseudo-real case.

Proof: For a self-conjugate Rep; $M(G)$, there exists a transform

$$X^{-1} M(g) X = M^*(g)$$, where X is a unitary transformati

Then $M(g) = (X^*)^{-1} M^*(g) X^* = X^T (X^{-1} M(g) X) X^*$

$$= X^T X^{-1} M(g) (X^T X^{-1})^{-1}$$

From Schul's lemma, we have $X^T X^{-1} = c\mathbb{1}$, $\Rightarrow X^T = cX$

$$X = (X^T)^T = \tau X^T = \tau^2 X \Rightarrow \tau = \pm 1.$$

equals zero if M^* is non-equivalent to M.

Consider $$\sum_{g\in G} \chi(g^2) = \sum_{\mu\nu} \sum_{g\in G} \left(M^*_{\mu\nu}(g)\right)^* M_{\nu\mu}(g)$$

$$= \sum_{\mu\nu} \sum_{g\in G} \{X^{-1} M(g) X\}^*_{\mu\nu} M_{\nu\mu}(g) = \sum_{\mu\nu, g\in G} X^T_{\mu\rho} M^*_{\rho\lambda}(g) \tau X^{T,*}_{\lambda\nu} M_{\nu\mu}(g)$$

$$= \sum_{\mu\nu, g\in G} \tau X_{\rho\mu} M^*_{\rho\lambda}(g) (X^{-1})_{\lambda\nu} M_{\nu\mu}(g)$$

$$\sum_{g\in G} M^*_{\rho\lambda}(g) M_{\nu\mu}(g) = \frac{|G|}{n_m} \delta_{\rho\nu} \delta_{\lambda\mu} \quad \text{if } M^* \sim M$$

$$\Rightarrow \sum_{g\in G} \chi(g^2) = \begin{cases} 0 & \text{if } M^* \nsim M \\ \dfrac{|G|\tau}{n_m} \sum_{\nu\lambda} X_{\nu\lambda} (X^{-1})_{\lambda\nu} = |G|\tau, & \text{if } M^* \sim M \end{cases}$$

Then we can see that actually τ is independent on representation.

① If $M(G)$ is a real representation, we can choose $X = I$ and $M(g)$ to be real, then $\tau = 1$. And if $\tau = 1$, the X is a symmetric unitary matrix, it can be proved that X can be represented as $X = Y^2$ where Y is also a symmetric unitary matrix. Then

please check!

$$Y^{-2} D(R) Y^2 = D^*(R) \Rightarrow Y^{-1} D(R) Y = Y D^*(R) Y^{-1} = \left(Y^{-1} D(R) Y\right)^*$$

(Y satisfy $Y^* = Y^{-1}$). Then $Y^{-1} D(R) Y$ is a real representation.

② If $M(G)$ cannot be formulated as a real representation $\Rightarrow \tau = -1$. otherwise, we have contridication.

Hence, we have the characters of representations of finite groups satisfy

$$\boxed{\frac{1}{|G|}\sum_{g\in G}\chi(g^2) = \begin{cases} 0, & \text{non-self conjugate Rep} \\ 1 & \text{real} \\ -1 & \text{pseudo-real} \end{cases}}$$

For Abeliean group.

① cyclic group: Z_N.

if $N = 2n$, the identity Rep and the one with angular momentum $m = n$ are real. Other representations are complex.

② An example of pseudo-real : spin-1/2 representation of $SU(2)$

$$\mathcal{U} = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}$$ with $|a|^2 + |b|^2 = 1$. $\mathcal{U}$ is parameterized by 3 real numbers: $\mathcal{U} = e^{-i\frac{\sigma_z}{2}\alpha} e^{-i\frac{\sigma_y}{2}\beta} e^{-i\frac{\sigma_z}{2}\gamma} \leftarrow$ Eulerian angles

$$\mathcal{U}^* = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Hence $\mathcal{U}^* \sim \mathcal{U}$. But it's impossible to find a basis that a and b are real, otherwise, we can parameterize $\mathcal{U}$ by 2 real numbers.