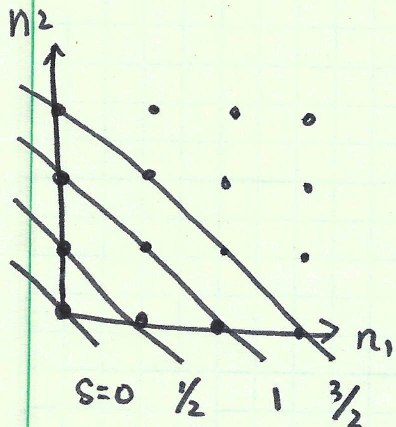
Lect 4 Representations of SU(2) — Wigner D-matrix

The 2D harmonic oscillator has an SU(2) symmetry

$$H = \hbar\omega\,[a_1^\dagger a_1 + a_2^\dagger a_2] \quad \text{where} \quad a_i = \frac{1}{\sqrt{2}}\left(\frac{x_i}{l} + i\, P_i \frac{l}{\hbar}\right)$$



① define $J_z = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)$

$J_+ = a_1^\dagger a_2, \quad J_- = a_2^\dagger a_1$

then $[J_i, J_j] = i\epsilon_{ijk} J_k$, — Schwinger boson Representation of SU(2) algebra.

② $a_1^\dagger a_1 + a_2^\dagger a_2 = 2S$ is conseverd, which specify each representation.

Let us check each energy level

1) $S=0$: the ground state $|\Omega\rangle$, $a_i|\Omega\rangle = 0$. — trivial Rep.

2) $S = 1/2 \quad |i\rangle = a_i^\dagger |\Omega\rangle$ — fundamental Rep

$$\hat{\mathcal{U}}(R) = e^{-iJ_z\alpha}\, e^{-iJ_y\beta}\, e^{-iJ_z\gamma}$$

For $S = 1/2 \quad \mathcal{U}(R) = e^{-i\frac{\sigma_z}{2}\alpha}\, e^{-i\frac{\sigma_y}{2}\beta}\, e^{-i\frac{\sigma_z}{2}\gamma}$

and $\boxed{\langle i|\hat{\mathcal{U}}(R)|j\rangle = \mathcal{U}_{ij}(R)}$.

How $a_i^\dagger$ transform under $\hat{\mathcal{U}}(R)$?

$$\mathcal{U}(R)|i\rangle = \sum_j |j\rangle\, \mathcal{U}_{ji}(R) \Rightarrow \mathcal{U}(R)\, a_i^\dagger\, \mathcal{U}^{-1}(R)\, |\Omega\rangle$$
$$= \sum_j a_j^\dagger\, \mathcal{U}_{ji}(R)\, |\Omega\rangle$$

$$\Rightarrow \quad \hat{U}(R)\, a_i^{\dagger}\, \hat{U}^{\dagger}(R) = \sum_j a_j^{\dagger}\, U_{ji} \rightarrow \text{Representation } 2\times 2 \text{ matrix}$$

Rotation operator in terms of $a_i^{\dagger}, a_i$,

For example, for $\hat{U}(R) = e^{-iJ_y\beta}$, we have

$$e^{-iJ_y\beta}[a_1^{\dagger} \;\; a_2^{\dagger}]\, e^{iJ_y\beta} = [a_1^{\dagger} \;\; a_2^{\dagger}]\, e^{-i\frac{\sigma_y}{2}\beta} = [a_1^{\dagger} \;\; a_2^{\dagger}] \begin{bmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\beta/2 & \cos\frac{\beta}{2} \end{bmatrix}$$

i.e.

$$\begin{cases} e^{-iJ_y\beta}\, a_1^{\dagger}\, e^{iJ_y\beta} = a_1^{\dagger}\cos\frac{\beta}{2} + a_2^{\dagger}\sin\frac{\beta}{2} \\ e^{-iJ_y\beta}\, a_2^{\dagger}\, e^{iJ_y\beta} = -a_1^{\dagger}\sin\frac{\beta}{2} + a_2^{\dagger}\cos\frac{\beta}{2} \end{cases}$$

HW: Directly prove the above results from operator calculation by using $J_y = \frac{1}{2i}(a_1^{\dagger}a_2 - a_2^{\dagger}a_1)$, $[J_y\, a_1^{\dagger}] = -\frac{1}{2i}a_2^{\dagger}$, $[J_y\, a_2^{\dagger}] = \frac{1}{2i}a_1^{\dagger}$

③ For a general energy level $a_1^{\dagger}a_1 + a_2^{\dagger}a_2 = 2S$, we can label the state

$$|j\,m\rangle = \frac{a_1^{\dagger\, j+m}\, a_2^{\dagger\, j-m}}{\sqrt{(j+m)!(j-m)!}}\, |\Omega\rangle ,$$

$$\langle jm'|\, \hat{U}(R)\, |jm\rangle = D^j_{m'm}(R) = e^{-im'\alpha - im\gamma}\, d^j_{m'm}(\beta)$$

$$\langle jm'|\, e^{-iJ_z\alpha}\, e^{-iJ_y\beta}\, e^{-iJ_z\gamma}\, |jm\rangle = e^{-im'\alpha - im\gamma}\, \langle jm'|\, e^{-iJ_y\beta}\, |jm\rangle$$

$$\boxed{\text{where } d^j_{m'm}(\beta) = \langle jm'|\, e^{-iJ_y\beta}\, |jm\rangle}$$

$$e^{-iJ_y\beta}|jm\rangle = \frac{1}{\sqrt{(j+m)!(j-m)!}}\left(a_1^\dagger\cos\frac{\beta}{2}+a_2^\dagger\sin\frac{\beta}{2}\right)^{j+m}\left(-a_1^\dagger\sin\frac{\beta}{2}+a_2^\dagger\cos\frac{\beta}{2}\right)^{j-m}|\Omega\rangle$$

$$= \frac{1}{\sqrt{(j+m)!(j-m)!}}\sum_{m'=-j}^{j}\sum_{\sigma}\binom{j+m}{j-m'-\sigma}\binom{j-m}{\sigma}\left(a_1^\dagger\cos\frac{\beta}{2}\right)^{m+m'+\sigma}\left(a_2^\dagger\sin\frac{\beta}{2}\right)^{j-m'-\sigma}$$

$$(-)^{j-m-\sigma}\left(a_1^\dagger\sin\frac{\beta}{2}\right)^{j-m-\sigma}\left(a_2^\dagger\cos\frac{\beta}{2}\right)^{\sigma}|\Omega\rangle$$

$$= \frac{1}{\sqrt{(j+m)!(j-m)!}}\sum_{m'=-j}^{j}\sum_{\sigma}\binom{j+m}{j-m'-\sigma}\binom{j-m}{\sigma}(a_1^\dagger)^{j+m'}(a_2^\dagger)^{j-m'}$$

$$(-)^{j-m-\sigma}\cdot\left(\cos\frac{\beta}{2}\right)^{m+m'+2\sigma}\left(\sin\frac{\beta}{2}\right)^{2j-2\sigma-m'-m}|\Omega\rangle$$

$$\left.\begin{array}{l} 0\le\sigma\le j-m \\ -m-m'\le\sigma\le j-m' \end{array}\right\}\Rightarrow \max(0,-m-m')\le\sigma\le\min(j-m,j-m')$$

$$|jm'\rangle = \frac{1}{\sqrt{(j+m')!(j-m')!}}(a_1^\dagger)^{j+m'}(a_2^\dagger)^{j-m'}|\Omega\rangle$$

$$\Rightarrow d^j_{m'm} = \frac{\sqrt{(j+m')!(j-m')!}}{\sqrt{(j+m)!(j-m)!}}\sum_\sigma\binom{j+m}{j-m'-\sigma}\binom{j-m}{\sigma}(-)^{j-m-\sigma}$$

$$\left(\cos\frac{\beta}{2}\right)^{2\sigma+m+m'}\left(\sin\frac{\beta}{2}\right)^{2j-2\sigma-m-m'}$$

$$= \sqrt{\frac{(j+m)!(j-m)!}{(j+m')!(j-m')!}}\left(\cos\frac{\beta}{2}\right)^{m+m'}\left(\sin\frac{\beta}{2}\right)^{m-m'}\left\{\sum_\sigma\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}\cdot\right.$$

$$\left.(-)^{j-m-\sigma}\binom{j+m}{j-m'-\sigma}\binom{j-m}{\sigma}\left(\cos\frac{\beta}{2}\right)^{2\sigma+m+m'}\left(\sin\frac{\beta}{2}\right)^{2j-2\sigma-m-m'}\right\}$$

It can be represented in terms of Jacobi polynomial

$$P_n^{(\alpha,\beta)}(x) = \frac{(-)^n}{2^n n!} (1-x)^{-\alpha}(1+x)^{-\beta} \frac{d^n}{dx^n}\left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right)$$

HW: Prove that

① $$P_n^{(\alpha,\beta)}(x) = \sum_{\ell=0}^{n} (-)^{n+\ell} \binom{\alpha+n}{\ell}\binom{\beta+n}{n-\ell}\left(\frac{1-x}{2}\right)^{n-\ell}\left(\frac{1+x}{2}\right)^{n-\ell}$$

and then $$d^j_{m'm}(\beta) = \sqrt{\frac{(j+m)!\,(j-m)!}{(j+m')!\,(j-m')!}}\left(\cos\frac{\beta}{2}\right)^{m+m'}\left(\sin\frac{\beta}{2}\right)^{m-m'} P_{j-m}^{m-m',\,m+m'}(\cos\beta)$$

HW 2: Prove the following properties of D-matrices

① $$d^j_{m'm}(\beta) = (-)^{m'-m} d^j_{mm'}(\beta) = (-)^{m'-m} d^j_{-m',-m}(\beta)$$

② $$d^\ell_{0m}(\beta) = \left[\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(x) -$$

where $P_\ell^m(x)$ is the associated Lengdre polynomial. You can use the follwing results.

$$P_\ell^m(x) = \frac{1}{2^\ell \ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell = \frac{(\ell+m)!}{2^m \ell!}(1-x^2)^{\frac{m}{2}} P^{m,m}_{\ell-m}(x)$$

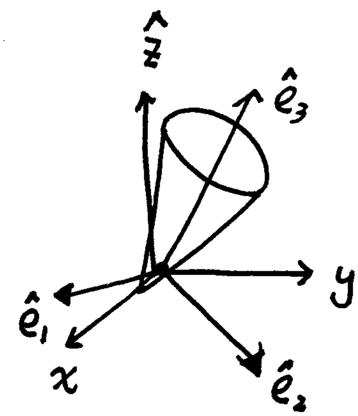
③ $$D^\ell_{00}(\alpha\beta\gamma) = d^\ell_{00}(\beta) = P_\ell(\cos\beta)$$

$$D^\ell_{0m}(\alpha\beta\gamma) = e^{-im\gamma} d^\ell_{0m}(\beta) = (-)^m\left(\frac{4\pi}{2\ell+1}\right)^{1/2} Y^*_{\ell m}(\beta,\gamma)$$

Lect 1 D-matrix as rotation wavefunctions — spinning top

Let us consider a rigid rotor, how to describe its rotation wavefunction in a quantum mechanical way? Physically, this rotor can be a molecule. Now, we are quantizing the motion of a top.

The configuration space of a top can be denoted by the Eulerian angles (α, β, γ).



The rolation between the body frame $(e_1 e_2 e_3)$ and the lab frame $(x\, y\, z)$ is

$$(\hat{e}_1, \hat{e}_2, \hat{e}_3) = (x, y, z) \underbrace{\begin{bmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma, & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma, & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma, & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma, & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma, & \sin\beta\sin\gamma, & \cos\beta \end{bmatrix}}_{T(\alpha\beta\gamma)=}$$

SO(3) matrix. — special orthoganal matrix.

Ex: prove the above relation following the definition of Eulerian angles (α, β, γ). Hint: Check the rotation matrice for x, y, z-axes, and perform the matrice product.

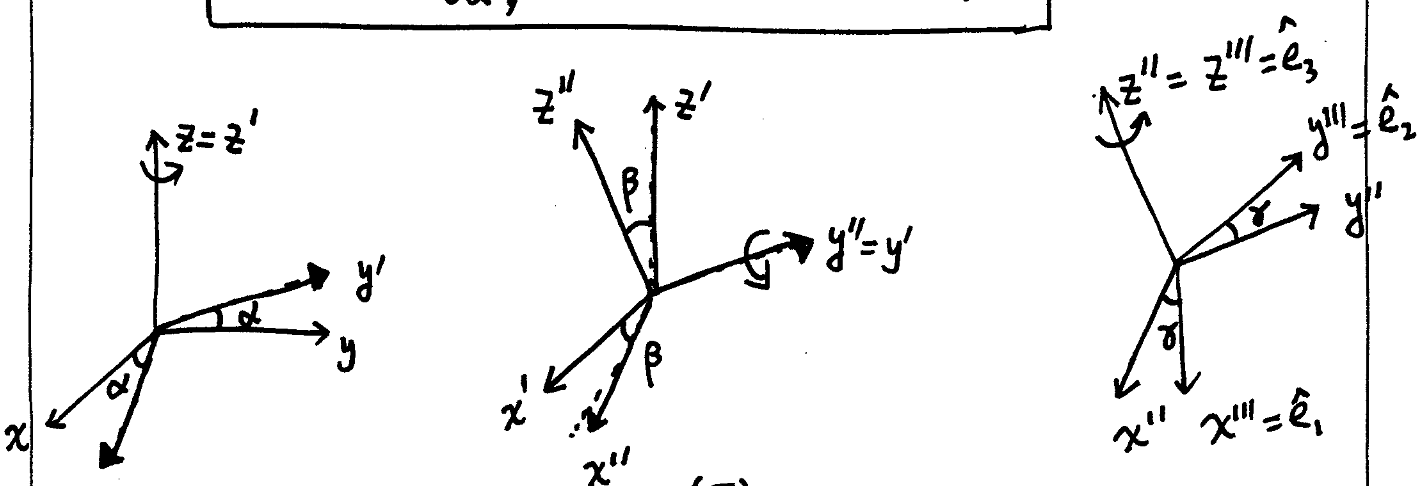
~~The angular momenta in the Lab frame is defined as~~

L_x, L_y, L_z, which are conserved quantities, and we will prove it later. Q_1, Q_2, Q_3 are angular momentum components projections on e_1, e_2 and e_3 axes. We know that from classic mechanics

$$H = \frac{Q_1^2}{2I_1} + \frac{Q_2^2}{2I_2} + \frac{Q_3^2}{2I_3}. \quad \text{(a free top, no external torque).}$$

Next we derive the expressions of $L_{x,y,z}$ and $Q_{1,2,3}$ in terms of α, β, γ. According to the definition,

$$\boxed{\hat{L}_z = -i\hbar\frac{\partial}{\partial\alpha}, \qquad \hat{Q}_3 = -i\hbar\frac{\partial}{\partial\gamma}}$$



(I) (II) (III)

According to (II), $\hat{L}_{y'} = -i\hbar\frac{\partial}{\partial\beta}$, and $\hat{y}' = \cos\alpha\,\hat{y} - \sin\alpha\,\hat{x}$

$$\Rightarrow \quad \cos\alpha\,\hat{L}_y - \sin\alpha\,\hat{L}_x = \hat{L}_{y'} = -i\hbar\frac{\partial}{\partial\beta} \qquad (*)$$

$$\hat{z}'' = \cos\beta\,\hat{z} + \sin\beta\,\hat{x}' = \cos\beta\,\hat{z} + \sin\beta(\cos\alpha\,\hat{x} + \sin\alpha\,\hat{y})$$

$$\Rightarrow \hat{L}_{z''} = \hat{Q}_3 = -i\hbar\frac{\partial}{\partial\gamma} = \cos\beta\,\hat{L}_z + \sin\beta\cos\alpha\,\hat{L}_x + \sin\beta\sin\alpha\,\hat{L}_y$$

$$\Rightarrow \quad \cos\alpha\,\hat{L}_x + \sin\alpha\,\hat{L}_y = -\frac{i\hbar}{\sin\beta}\frac{\partial}{\partial\gamma} + \cot\beta\; i\hbar\frac{\partial}{\partial\alpha} \qquad (**)$$

from (*) and (**), we arrive at

$$\boxed{\begin{aligned}\hat{L}_x &= -i\hbar\left[-\cos\alpha\cot\beta\frac{\partial}{\partial\alpha} - \sin\alpha\frac{\partial}{\partial\beta} + \frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right]\\ \hat{L}_y &= -i\hbar\left[-\sin\alpha\cot\beta\frac{\partial}{\partial\alpha} + \cos\alpha\frac{\partial}{\partial\beta} + \frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right]\end{aligned}}$$

Similarly $\quad \hat{e}_1 = \hat{x}''' = \hat{x}''\cos\gamma + \hat{y}''\sin\gamma = (\cos\beta\,x' - \sin\beta\,z')\cos\gamma + \hat{y}'\sin\gamma$

$$\Rightarrow \quad = \cos\beta\underset{\cos\gamma}{\wedge}[\cos\alpha\,\hat{x} + \sin\alpha\,\hat{y}] - \sin\beta\cos\gamma\,\hat{z}' + \hat{y}'\sin\gamma$$

$$\hat{Q}_1 = \cos\beta\underset{\cos\gamma}{\wedge}[\cos\alpha\,\hat{L}_x + \sin\alpha\,\hat{L}_y] - \sin\beta\cos\gamma\,\hat{L}_z - i\hbar\frac{\partial}{\partial\beta}\sin\gamma$$

$$\hat{Q}_1 = -i\hbar\left[\sin\gamma\frac{\partial}{\partial\beta} - \frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} + \cot\beta\cos\gamma\frac{\partial}{\partial\gamma}\right]$$

$$\& \quad \hat{e}_2 = \hat{y}''' = -\sin\gamma\,\hat{x}'' + \cos\gamma\,\hat{y}'$$

$$= -\sin\gamma\,[\cos\beta\,x' - \sin\beta\,z'] + \cos\gamma\,\hat{y}'$$

$$= -\sin\gamma\cos\beta\,[\cos\alpha\,\hat{x} + \sin\alpha\,\hat{y}] + \sin\gamma\sin\beta\,z' + \cos\gamma\,\hat{y}'$$

$$\Rightarrow \hat{Q}_2 = -\sin\gamma\cos\beta\left[\cos\alpha\,\hat{L}_x + \sin\alpha\,\hat{L}_y\right] + \sin\gamma\sin\beta\,\hat{L}_z - \cos\gamma\; i\hbar\frac{\partial}{\partial\beta}$$

$$\Rightarrow \boxed{\begin{aligned} Q_1 &= -i\hbar\left[-\frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} + \sin\gamma\frac{\partial}{\partial\beta} + \cot\beta\cos\gamma\frac{\partial}{\partial\gamma} \right] \\ Q_2 &= -i\hbar\left[\frac{\sin\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} + \cos\gamma\frac{\partial}{\partial\beta} - \cot\beta\sin\gamma\frac{\partial}{\partial\gamma} \right] \end{aligned}}$$

Ex: check that $Q_i = (\hat{e}_i \cdot \vec{L}) = \hat{x}_j T_{ji}(\alpha\beta\gamma) \cdot \hat{\vec{L}}$

$$\Rightarrow \boxed{Q_i = T_{ij}(\alpha\beta\gamma)\,\hat{L}_j}$$

Since $Q_i = (\hat{e}_i \cdot \vec{L})$, under rotations, both $\hat{e}_i$ and $\vec{L}$ transform in the same way and thus keep the inner product invariant. In other words, Q_i is a scalar under rotation. $\Rightarrow$

$$D^\dagger(g)\, Q_i\, D(g) = Q_i, \quad \text{or} \quad e^{i\vec{L}\cdot\hat{n}\theta}\, Q_i\, e^{-i\vec{L}\cdot\hat{n}\theta} = Q_i$$

$$\Rightarrow \boxed{[L_j, Q_i] = 0.}$$

The Lab frame angular momentum and body frame ones commute!

Ex: please check $[L_j, Q_i] = 0$ from their expressions using (α, β, γ) by brutal force calculation.

For a free top, we have $[L^2, H] = [L_z, H] = [L^2, L_z] = 0$, thus we can chose $(L^2\ L_z)$ to characterize the rotation eigenstate of a top. However, we know that (L^2, L_z) is complete for a point particle confinied on a sphere, but a top is more than a single particle. Later, we will see that we need an extra quantum number.

Now let use denote the wavefunction $\psi_{IM}(g)$, where IM are the quantum numbers for L^2, L_z, respectively. The configuration space of a top is characterized by (α, β, γ), which determines an SO(3) $\overbrace{\text{rotation, i.e. the rotation from the}}^{g(\alpha\beta\gamma)}$ Lab frame (xyz) to the body frame (e_1, e_2, e_3). Thus the configuration space of a top is the same of the group space of SO(3). We will use "g" as the coordinate of rotating top.

Let us apply the rotation g_0 on the wavefunction $\psi_{Im}(g)$.

$$\underset{\substack{\uparrow \\ \text{rotation}}}{R(g_0)}\ \psi_{Im}(\underset{\substack{\nwarrow \\ \text{coordinate} \\ \text{of the configuration space of top}}}{g}) = \psi_{IM}(g_0^{-1} g)$$

$\nwarrow$ product according to group operation

on the other hand, $R(g_0)\,\psi_{Im}(g) = \sum_{M'} \psi_{IM'}(g)\, D^I_{M'M}(g_0)$.

Let set $g_0 = g \Rightarrow \psi_{IM}(e) = \sum_{M'} \psi_{IM'}(g)\, D^I_{M'M}(g) = \sum_{M'} (D^I_{MM'})^T\, \psi_{IM'}(g)$

$$\boxed{\psi_{IM}(g) = \sum_{M'} \left(D^{I,T}\right)^{\dagger}_{MM'}(g)\ \psi_{IM'}(e) = \sum_{M'} D^{*,I}_{M,M'}(g)\,\psi_{IM'}(e)}$$

Before we move on, let's define the ~~orthagonal~~ orthagnoal conditions of $D^{*I}_{MM'}(g)$.

① measure $\int dg = \int_0^{2\pi} d\alpha \underbrace{\int_0^{\pi} d\beta}_{\sin\beta} \int_0^{2\pi} d\gamma = 8\pi^2$

$D^{I_1}_{m_1' m_1}(g) = \langle I_1 m_1' | D(g) | I_1 m_1 \rangle$, $\quad D^{I_2}_{m_2' m_2}(g) = \langle I_2 m_2' | D(g) | I_2 m_2 \rangle$

Theorem: $\int dg\, D^{I_1 *}_{m_1' m_1}(g)\, D^{I_2}_{m_2' m_2}(g) = C(I_1)\,\delta_{I_1 I_2}\,\delta_{m_1' m_2'}\,\delta_{m_1 m_2}$, and

$C(I_1) = \dfrac{8\pi^2}{2I_1+1}$.

Proof: define an operator $P = \int dg\, D^{\dagger}(g)\, |I_1 m_1'\rangle\langle I_2 m_2'|\, D(g)$

where $D(g) = e^{-i\vec{J}\cdot\hat{n}\theta}$.

Let us define a rotation g_0, and $P \rightarrow$ ~~$g_0 P g_0^{-1}$~~ $P' = D^{\dagger}(g_0)\, P\, D(g_0)$

~~$g_0 P g_0^{-1} = \int dg\, g$~~

$$P' = \int dg\, D^{\dagger}(g_0) D^{\dagger}(g)\, |I_1 m_1'\rangle\langle I_2 m_2'|\, D(g) D(g_0)$$

$$= \int dg\, D^{\dagger}(g g_0)\, |I_1 m_1'\rangle\langle I_2 m_2'|\, D(g g_0)$$

$$= \int d(g' g_0)\, D^{\dagger}(g')\, |I_1 m_1'\rangle\langle I_2 m_2'|\, D(g') = \int dg\, D^{\dagger}(g)\, |I_1 m'\rangle\langle I_2 m_2'|\, D(g)$$

measure is invariant

$= P$

then $\int dg \, D^{I_1 *}_{m_1' m_1}(g) \, D^{I_2}_{m_2' m_2}(g) = \langle I_1 m_1 | P | I_2 m_2 \rangle$

since P is rotationally invariant $\Rightarrow \int dg \, D^{I_1^*}_{m_1' m_1}(g) \, D^{I_2}_{m_2' m_2}(g) \propto \delta_{I_1 I_2} \delta_{m_1 m_2}$

Similarly $\Rightarrow \qquad \propto \delta_{m_1' m_2'}$

The result is also only dependent on I_1, but not m's. i.e. $C(I_1)$

$$\Rightarrow \quad C(I_1) = \frac{1}{2I_1+1} \sum_{m_1} \int D^{* I_1}_{m_1' m_1}(g) \, D^{I_1}_{m_1' m_1}(g) \, dg$$

$$= \frac{1}{2I_1+1} \sum_{m_1} \int \left[D^{I_1}_{m_1 m_1'}(g) \right]^{\dagger} D^{I_1}_{m_1' m_1}(g) \, dg = \frac{1}{2I_1+1} \int dg = \frac{8\pi^2}{2I_1+1}$$

$$\Rightarrow \boxed{\int D^{I_1^*}_{m_1' m_1}(g) \, D^{I_2}_{m_2' m_2}(g) \, dg = \frac{8\pi^2}{2I_1+1} \delta_{I_1 I_2} \delta_{m_1' m_1} \delta_{m_2' m_2}}$$

Now we can interpret $D^{*, I}_{M M'}(g)$ as the basis of rotation functions of a top. We also know that IM are not complete to describe top. We can assign M' as another quantum number to classify bases.

orth-normal basis (IMM')

$$\boxed{\psi_{IMM'}(g) = \sqrt{\frac{2I+1}{8\pi^2}} \, D^{*, I}_{M M'}(g)}$$

Next, we need to figure out what is the physical meaning of the quantum number of M'.

~~by definition, we apply rotation on top wavefunction~~

$$e^{-i\theta\hat{n}\cdot\vec{J}}\ \psi(g) = \psi(g^{-1}(\hat{n},\theta)\cdot g).$$

Replace $\psi(g) = D^{I*}_{MM'}(\alpha\beta,\gamma)$, where $g = g(\alpha\beta\gamma) \Rightarrow$

$$e^{-i\theta\hat{n}\cdot\vec{J}}\ D^{I*}_{MM'}(\alpha\beta\gamma) = \left[D^{I}_{MM'}(g_0^{-1}(\hat{n},\theta)g)\right]^*$$

$$= \sum_{M''} D^{I,*}_{MM''}(g_0^{-1}(\hat{n},\theta))\ D^{I,*}_{M''M'}(g) = \sum_{M''} D^{I,*}_{M''M'}(g)\ D^{I}_{M'',M}(g_0(\hat{n},\theta))$$

$\longrightarrow$ take infinitesimal rotation, and remember $g = g(\alpha\beta\gamma)$

$$\boxed{\hat{n}\cdot\vec{J}\ D^{I,*}_{MM'}(\alpha\beta\gamma) = \sum_{M''} D^{I,*}_{M'',M'}(\alpha\beta\gamma)\ \langle IM''|\,\hat{n}\cdot\vec{J}\,|IM\rangle}$$

operation on the first m-index.

now take $\hat{n}$ to be one of the top body axes, $\hat{e}_{k=1,2,3}$

$$\hat{e}_K = g(\alpha\beta\gamma)\ \hat{i}_k. \qquad \hat{i}_k = \hat{x}, \hat{y}, \hat{z};\ \text{the fixed frame axis.}$$

$$Q_K = \hat{e}_K\cdot\vec{J} \Rightarrow Q_K D^{I,*}_{MM'}(\alpha\beta\gamma) = \sum_{M''} D^{I,*}_{M''M'}(\alpha\beta\gamma)\langle IM''|\,\hat{e}_K\cdot\vec{J}\,|IM\rangle$$

$\Rightarrow$

$$\hat{e}_K\cdot\vec{J} = \vec{J}\cdot g(\alpha\beta\gamma)\hat{i}_K = (g^{-1}\vec{J})\cdot\hat{i}_K = D(g)\left(\vec{J}\cdot\hat{i}_K\right)D^{-1}(g)$$

$$\Rightarrow \langle IM''|\,Q_K\,|IM\rangle = \sum_{K'K''} \langle IM''|D(g)|IK'\rangle\langle IK'|\,\vec{J}\cdot\hat{i}_K\,|IK''\rangle\langle IK''|D^{-1}(g)|IM\rangle$$

$$= \sum_{K',K''} D^{I}_{M''K'}(g)\ D^{I}_{K''M}(g^{-1})\ \langle IK'|\,J_K\,|IK''\rangle$$

$\Rightarrow$

$$Q_K\, D^{I,*}_{MM'}(\alpha\beta\gamma) = \sum_{\substack{M'' \\ K'K''}} D^{I,*}_{M''M'}(\alpha\beta\gamma)\, D^{I}_{M'',K'}(g)\, D^{I}_{K''M}(g^{-1})\, \langle IK' | J_k | IK'' \rangle$$

Sum over M''

$$\sum_{M''} D^{I,*}_{M''M'}(\alpha\beta\gamma)\, D^{I}_{M''K'}(\alpha\beta\gamma) = \delta_{K'M'}$$

$$\Rightarrow Q_K\, D^{I,*}_{MM'}(\alpha\beta\gamma) = \sum_{K''} D^{I}_{K''M}(g^{-1})\, \langle IM' | J_k | IK'' \rangle$$

$$= \sum_{K''} D^{I,*}_{MK''}(g)\, \langle IM' | J_k | IK'' \rangle$$

set $k=3$ $\Rightarrow$ $\langle IM' | J_z | IK'' \rangle = \delta_{M'K''}\, M'$.

$\Rightarrow$ $\boxed{Q_3\, D^{I,*}_{MM'} = M'\, D^{I,*}_{MM'}}$ $\leftarrow$ M' is the eigen-number of Q_3.

more generally $\Rightarrow$

$$\boxed{(\vec{J}\cdot\hat{e}_k)\, D^{I,*}_{MM'}(\alpha\beta\gamma) = \sum_{M''} D^{I,*}_{MM''}(\alpha\beta\gamma)\, \langle IM' | \vec{J}\cdot\hat{e}_k | IM'' \rangle}$$

operations on ~~the second index in the lower line~~ the second m-index.