Solution to HW 3

1. a) $V(q) = \int d^3\vec{r}\ e^{-i\vec{q}\cdot\vec{r}}\ V(r) = \int d^3\vec{r}\ e^{-i\vec{q}\cdot\vec{r}}\ \frac{g}{4\pi r}\ e^{-r/a} = \frac{g}{q^2 + (1/a)^2}$

the Hartree - Fock correction

$$E = E_{free} + \frac{1}{2V} \sum_{kk',\sigma\sigma'} \{ V(0) - \delta_{\sigma\sigma'} V(\vec{k}-\vec{k}') \} n_{k\sigma} n_{k'\sigma'}$$

Thus $f_{k\sigma,k'\sigma'} = \left(V(0) - \delta_{\sigma\sigma'} V(\vec{k}-\vec{k}') \right] \cdot \frac{1}{V} \leftarrow$ volume

In order to get Landau parameters. we need first find density of states, i.e., N(0), which is related to v_F.

$$\delta \mathcal{E}^{HF}(k,\sigma) = V(0)\cdot n - \int_0^{k_F} \frac{d^3\vec{k}'}{(2\pi)^3} \frac{g}{|\vec{k}-\vec{k}'|^2 + (1/a)^2}$$

$$= V(0)\, n - \frac{g}{4\pi^2} \int_0^{k_F} k'^2 dk' \int_{-1}^{1} dx \frac{1}{k^2 + k'^2 + (1/a)^2 - 2kk'x}$$

the integral $\int_0^{k_F} k'^2 dk' \left(-\frac{1}{2kk'}\right) \ln \frac{(k+k')^2 + (1/a)^2}{(k-k')^2 + (1/a)^2}$

Check integral table or by mathimetica

$$\int dk'\ k' \ln\left[(k+k')^2 + (1/a)^2\right] = \left(k - \frac{k'}{2}\right) k' - \frac{2k}{a} \operatorname{arctg}(k+k')a$$

$$+ \frac{k'^2 + (1/a)^2 - k^2}{2} \ln\left[(k+k')^2 + (1/a)^2\right]$$

$$\Rightarrow \int dk'\, k' \ln \frac{(k+k')^2+(1/a)^2}{(k-k')^2+(1/a)^2} = 2kk' - \frac{2k}{a}\{\text{arctg}\,(k'+k)a + \text{arctg}\,(k'-k)a\}$$

$$+ \frac{k'^2+(1/a)^2-k^2}{2} \ln \frac{(k'+k)^2+(1/a)^2}{(k'-k)^2+(1/a)^2}$$

$$\int_0^{k_f} dk'\, k' \ln \frac{(k+k')^2+(1/a)^2}{(k-k')^2+(1/a)^2} = 2k_f k - \frac{2k}{a}\{\text{arctg}\,(k_f+k)a + \text{arctg}\,(k_f-k)a\}$$

$$+ \frac{k_f^2-k^2+(1/a)^2}{2} \ln \frac{(k_f+k)^2+(1/a)^2}{(k_f-k)^2+(1/a)^2}$$

$$\delta\mathcal{E}^{HF}(k) = V(0)\cdot n - \frac{g}{4\pi^2}\Big[k_f - \frac{1}{a}[\text{arctg}\,(k_f+k)a + \text{arctg}\,(k_f-k)a]$$

$$+ \frac{k_f^2-k^2+(1/a)^2}{4k} \ln \frac{(k_f+k)^2+(1/a)^2}{(k_f-k)^2+(1/a)^2}$$

set $y = k/k_f$

$$\delta\mathcal{E}^{HF}(k_f\cdot y) = V(0)\cdot n - \frac{g}{4\pi^2}k_f\Big[1 - \frac{1}{k_f a}\big(\text{arctg}\,(k_f a(1+y)) + \text{arctg}\,(k_f a(1-y))\big)\Big]$$

$$+ \frac{1-y^2+(\frac{1}{k_f a})^2}{4y} \ln \frac{(1+y)^2+(\frac{1}{k_f a})^2}{(1-y)^2+(1/k_f a)^2}$$

$$v_F = \frac{\partial \mathcal{E}^{HF}}{\partial k} = \frac{k_f}{m} - \frac{gk_f}{4\pi^2}\frac{1}{k_f}\Big[-\frac{1}{k_f a}\Big(\frac{k_f a}{1+(k_f a)^2(1+y)^2} - \frac{k_f a}{1+(k_f a)^2(1-y)^2}\Big]$$

$$+ \Big(\frac{1+y^2+(\frac{1}{k_f a})^2}{-4y^2} \ln \frac{(1+y)^2+(\frac{1}{k_f a})^2}{(1-y)^2+(\frac{1}{k_f a})^2}\Big) + \frac{1-y^2+(\frac{1}{k_f a})^2}{4y}\Big(\frac{2(y+1)}{(1+y)^2+(\frac{1}{k_f a})^2} - \frac{2(y-1)}{(1-y)^2+(\frac{1}{k_f a})^2}\Big)$$

(set $y=1$)

$$v_F = \frac{k_f}{m} - \frac{g}{4\pi^2}\left\{ -\left[\frac{1}{1+4k_f^2a^2} - 1\right] + \frac{2+(1/k_fa)^2}{-4}\ln\left(1+4(k_fa)^2\right]\right.$$

$$\left. + \frac{(1/k_fa)^2}{4}\left[\frac{4}{4+1/(k_fa)^2}\right]\right\}$$

$$= \frac{k_f}{m} - \frac{g}{4\pi^2}\left[1 - \left(\frac{1}{2} + \frac{1}{4(k_fa)^2}\right)\ln\left(1+4(k_fa)^2\right)\right]$$

Expand it interms of power of k_fa to $O(k_fa)^4$

$$\Rightarrow \quad v_F = \frac{k_f}{m} + \frac{g}{3\pi^2}(k_fa)^4$$

$$\Rightarrow N(0) = N_{free}\cdot\left(\frac{v_F^0}{v_F}\right) = N_{free}\cdot\frac{\frac{k_f}{m}}{\frac{k_f}{m} + \frac{g}{3\pi^2}(k_fa)^4}$$

$$= N_{free}\frac{1}{1+\frac{m}{k_f}\cdot\frac{g}{3\pi^2}(k_fa)^4} \approx N_{free}\left[1 - \frac{gma}{3\pi^2}(k_fa)^3\right].$$

$$\Rightarrow \quad F^s_{kk'} = N(0)\, f^s_{kk'} = N(0)\left[V(0) - \frac{V}{2}(\vec{k}-\vec{k}')\right]$$

$$= N(0)\left[V(0) - \frac{1}{2}V\left(2k_f\sin\frac{\theta}{2}\right)\right]$$

$$F^a_{kk'} = N(0)\,(-1/2)\,V\left(2k_f\sin\frac{\theta}{2}\right)$$

θ is the angle between $k\,k'$.

$$F^{s,a}_\ell = \int_{-1}^{1} d\cos\theta\; F^{s,a}(\cos\theta)\,P_\ell(\cos\theta) \Big/ \int_{-1}^{1} d\cos\theta\,\left[P_\ell(\cos\theta)\right]^2$$

$$= \frac{2\ell+1}{2}\int_{-1}^{1} d\cos\theta\; F^{s,a}(\cos\theta)\,P_\ell(\cos\theta)$$

$$\int_{-1}^{1} d\cos\theta \ V(0)\, P_0(\cos\theta) = 2V(0) = 2ga^2$$

$$\int_{-1}^{1} d\cos\theta \ V(0)\, P_1(\cos\theta) = 0$$

$$\int_{-1}^{1} d\cos\theta \ V(2k_F|\sin\tfrac{\theta}{2}|)\, P_0(\cos\theta) = \int_{-1}^{1} dx \ \frac{g}{2k_f^2(1-x)+(1/a)^2} = \frac{g}{2k_f^2}\int_{-1}^{1} dx \ \frac{1}{1-x+\frac{1}{2k_f^2}(1/a)^2}$$

$$= \frac{g}{2k_f^2} \ln(1+4k_f^2a^2)$$

$$\int_{-1}^{1} d\cos\theta \ V(2k_f|\sin\tfrac{\theta}{2}|)\, P_1(\cos\theta) = \int_{-1}^{1} dx \ \frac{g\,x}{2k_f^2(1-x)+(1/a)^2} = \frac{g}{2k_f^2}\int_{-1}^{1} dx \ \left(-1 + \frac{1+\frac{1}{2k_f^2a^2}}{1-x+\frac{1}{2k_f^2a^2}}\right)$$

$$= \frac{g}{2k_f^2}\left[\left(1+\frac{1}{2k_f^2a^2}\right)\ln(1+4k_f^2a^2) - 2\right]$$

$$\Rightarrow \quad F_0^s = \frac{1}{2}\int_{-1}^{1} dx \ N(0)\left[V(0) - \frac{1}{2}V(2k_f|\sin\tfrac{\theta}{2}|)\right) = \frac{N(0)}{2}\left[\cdot 2ga^2 + \frac{-g}{4k_f^2}\ln(1+4k_f^2a^2)\right]$$

$$= \frac{1}{2}N_{free}\left(1 - \frac{g m a}{\pi^2}(k_f a)^3\right]\frac{g}{k_f^2}\left[-2(k_f a)^2 + (k_f a)^2 - \frac{1}{2}\cdot\frac{1}{4}(4k_f^2a^2)^2\right]$$

$$= \frac{1}{2}N_{free}\left(\frac{g}{k_f^2}\right)\left[(k_f a)^2 + 2(k_f a)^4\right] \qquad \text{where } \boxed{N_{free} = \frac{m k_f}{\pi^2\hbar^2}}.$$

$$F_0^a = \frac{1}{2}\int_{-1}^{1} dx \ N(0)\left[-\frac{1}{2}V(2k_f|\sin\tfrac{\theta}{2}|)\right] = \frac{1}{2}N(0)\,\frac{1}{2}\,\frac{g}{2k_f^2}\ln(1+4k_f^2a^2)$$

$$= \frac{1}{2}N_{free}\cdot\frac{-g}{k_f^2}\left[(k_f a)^2 - (k_f a)^4\right]$$

$$F_1^s = F_1^a = N(0)\,\frac{3}{2}\int_{-1}^{1} dx \ \left(-\frac{1}{2}V(2k_f\sin\tfrac{\theta}{2})\right)P_1(\cos\theta)$$

$$= \frac{3}{2}N_{free}\left(\frac{1}{2}\right)\left(\frac{g}{2k_f^2}\right)\left[\left(1+\frac{1}{2k_f^2a^2}\right)\left(4k_f^2a^2 - 8k_f^4a^4 + \frac{64}{3}k_f^6a^6\right) - 2\right]$$

$$= N_{free} \cdot \frac{-g}{k_f^2} (k_f a)^4 = \frac{-mga}{\pi^2} (k_f a)^3$$

b) $$\frac{C_v}{C_v\,free} = \frac{m^*}{m} = 1 + \frac{1}{3} F_1^s = 1 - \frac{1}{3} \frac{mga}{\pi^2} (k_f a)^3$$

$$\frac{\kappa}{\kappa_0} = \frac{m^*}{m} \frac{1}{1+F_0^s} = \left[1 - \frac{1}{3} \frac{mga}{\pi^2} (k_f a)^3\right] / \left(1 + \frac{gma}{2\pi^2}\left[(k_f a) + (k_f a)^3 2\right]\right)$$

$$\frac{\chi}{\chi_0} = \frac{m^*}{m} \frac{1}{1+F_0^a} = \left[1 - \frac{1}{3} \frac{mga}{\pi^2} (k_f a)^3\right] / \left\{1 - \frac{gma}{2\pi^2}\left[(k_f a) - 2(k_f a)^3\right]\right\}$$

c) $$V(r) = \frac{e^2}{r} e^{-k_{TF} r}$$ set $g = 4\pi e^2$, $a = \frac{1}{k_{TF}} = \frac{1}{\sqrt{\frac{4}{\pi} \frac{k_F}{a_0}}}$

$k_f a \sim \sqrt{k_f a_0} << 1$ a_0 is Bohr radius

$$\frac{gma}{2\pi^2\hbar^2} (k_f a) \simeq \frac{4\pi e^2}{2\pi^2 \hbar^2} m \left(\frac{4}{\pi} \frac{k_F}{a_0}\right)^{-1} k_f = -\frac{e^2 m a_0}{2\hbar^2} = -1/2$$

we neglect high order terms involving $(k_f a)^3$, $(k_f a)^4$, ...

$$\Rightarrow \frac{\kappa}{\kappa_0} = \frac{1}{1+1/2} \simeq \frac{2}{3}$$

$$\frac{\chi}{\chi_0} \simeq \frac{1}{1-1/2} \simeq 2$$

The concrete numbers here are not ~~important~~. The purpose here is to show a concrete exaple how interaction changes physical obeservables.

a).

2. From Boltzmann equation

$$\frac{\partial}{\partial t} n_p(r,t) + \nabla_p \mathcal{E}_p(r,t)\, \nabla_r n_p(r,t) - \nabla_r \mathcal{E}_p(r,t)\, \nabla_p n_p(r,t) = I[n_{p'}]$$

sum over momentum p

$$\frac{\partial}{\partial t} \int \frac{d^3p}{(2\pi)^3} n_p(r,t) + \nabla_r \cdot \int \frac{d^3p}{(2\pi)^3} \left(\nabla_p \mathcal{E}_p(r,t) \cdot n_p(r,t)\right) - \left[\int \frac{d^3p}{(2\pi)^3} \nabla_p \left(\nabla_r \mathcal{E}_p(r,t) n_p\right)\right]$$

$$= \int \frac{d^3p}{(2\pi)^3} I(n_p]$$

The 3rd term is an intergral of total divergence $\longrightarrow 0.$

The collision integral conserves particle numer $\rightarrow \int \frac{d^3p}{(2\pi)^3} I(n_p) = 0$

define
$$\boxed{\begin{aligned} n(r,t) &= \sum_\sigma \int \frac{d^3p}{(2\pi)^3} n_{p\sigma}(r,t), \\ \vec{j}(r,t) &= \sum_\sigma \int \frac{d^3p}{(2\pi)^3} \vec{\nabla}_p \mathcal{E}_{p\sigma}(r,t)\, n_{p\sigma}(r,t) \end{aligned}}$$

and we have $\frac{\partial}{\partial t} n(r,t) + \nabla_r j(r,t) = 0$

b) linearing the expression of $\vec{j}(r,t)$

$$\mathcal{E}_{p_\sigma}(r,t) = \mathcal{E}_p^0 + \int \frac{d^3p'}{(2\pi)^3} f^s_{\sigma\sigma'}(p,p')\, \delta n_{p'\sigma'}(r,t); \quad n_{p\sigma}^{(r,t)} = n_p^0 + \delta n_{p\sigma}(r,t)$$

$$\Rightarrow \vec{j}(r,t) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3} \nabla_p \mathcal{E}^0_{p_\sigma} \cdot \delta n_{p\sigma}(r,t) + \underbrace{\nabla_p \delta \mathcal{E}_{p_\sigma}(r,t) \cdot n_p^0}_{\text{partial deriavtive}}$$

$$= \sum_\sigma \int \frac{d^3p}{(2\pi)^3} \nabla_p \mathcal{E}_p^0\, \delta n_{p\sigma}(r,t) - \nabla_p n_p^0\, \delta \mathcal{E}_{p_\sigma}(r,t)$$

$$= \sum_\sigma \int \frac{d^3p}{(2\pi)^3} \, v_p \left[\delta n_{p\sigma}(r,t) - \frac{\partial n^0_{p\sigma}}{\partial \varepsilon_p} \cdot \int \frac{d^3p'}{(2\pi)^3} \, f_{\sigma\sigma'}(p,p') \, \delta n_{p\sigma'}(r,t) \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \, v_p \left[\delta n_p(r,t) \right] + \int \frac{d^3p}{(2\pi)^3} v_p \left(- \frac{\partial n^0_{p\sigma}}{\partial \varepsilon_p} \right) \int \frac{d^3p'}{(2\pi)^3} \, f^s(p,p') \, \delta n_{p'}(r,t)$$

$$\int \frac{d^3p}{(2\pi)^3} \, v_p \left(- \frac{\partial n^0_{p\sigma}}{\partial \varepsilon_p} \right) f^s(p,p') = N(0) \int \frac{d\Omega}{4\pi} \sum_l f^s_l \, P_l(\cos\theta) \, v_F \cdot \cos\theta \quad \left(\begin{array}{l} \text{set } p' \\ \text{along } z\text{-axis} \end{array} \right)$$

only $l=1$ term survives $\rightarrow \quad \frac{N(0)}{3} f^s_1 \vec{v}_{p'}$

$$\Rightarrow \vec{j}(r,t) = \int \frac{d^3p}{(2\pi)^3} \, \vec{v}_p \, \delta n_p(r,t) + \frac{F^s_1}{3} \int \frac{d^3p'}{(2\pi)^3} \, \vec{v}_{p'} \, \delta n_{p'}(r,t)$$

$$= \int \frac{d^3p}{(2\pi)^3} \, \vec{v}_p \left(1 + \frac{F^s_1}{3}\right) \delta n_p(r,t) = \int \frac{d^3p}{(2\pi)^3} \, \frac{\vec{p}}{m^*} \left(1 + \frac{F^s_1}{3}\right) \delta n_p(r,t)$$

c) on the other hand, by adiabatic continuity $\Rightarrow$

$$\vec{j}(r,t) = \int \frac{d^3p}{(2\pi)^3} \, \frac{\vec{p}}{m} \, \delta n_p(r,t) \quad \Rightarrow \quad \frac{1}{m} = \frac{1 + \frac{F^s_1}{3}}{m^*}$$

interactions don't change total momentum.

thus current doesn't change as interaction is slowly turn on.

d) For the spin case, we need to restore density-matrix structure of the distribution function

$$\underset{\alpha\beta}{n_p(r,t)} = \underset{\substack{\uparrow \\ \text{charge}}}{n_p(r,t)\, \delta_{\alpha\beta}} + \underset{\substack{\uparrow \\ \text{spin}}}{\vec{\sigma}_p(r,t)} \cdot \vec{\tau}_{\alpha\beta} \leftarrow \text{Pauli matrix}$$

Similarly, the quasi-particle energy can be written as

$$\mathcal{E}_{\alpha\beta}(p,r,t) = \mathcal{E}_p(r,t)\,\delta_{\alpha\beta} + \vec{h}_p(r,t)\cdot\vec{\tau}_{\alpha\beta}$$

The Boltzman equation changes to

$$\frac{\partial}{\partial t} n_p(r,t) + \frac{\partial}{\partial r}\left[\frac{\partial \mathcal{E}}{\partial p} n_p(r,t)\right] + \frac{\partial}{\partial p}\left[\frac{\partial \mathcal{E}}{\partial r} n_p(r,t)\right] - \frac{1}{i\hbar}\left[n_p(r,t)\ \mathcal{E}_p(r,t)\right]$$

$$= I_{coll}$$

after seperation of variables, we have

$$\frac{\partial n_p(r,t)}{\partial t} + \frac{\partial}{\partial r_i}\left[\frac{\partial \mathcal{E}}{\partial p_i} n_p + \frac{\partial \vec{h}_p}{\partial p_i}\cdot\vec{\sigma}_p\right] + \frac{\partial}{\partial p_i}\left[-\frac{\partial \mathcal{E}}{\partial r_i} n_p - \frac{\partial \vec{h}_p}{\partial r_i}\cdot\vec{\sigma}_p\right] = I_{coll\ charge}$$

$$\frac{\partial \vec{\sigma}_p(r,t)}{\partial t} + \frac{\partial}{\partial r_i}\left[\frac{\partial \mathcal{E}}{\partial p_i}\vec{\sigma}_p + \frac{\partial \vec{h}_p}{\partial p_i} n_p\right] + \frac{\partial}{\partial p_i}\left[-\frac{\partial \mathcal{E}}{\partial r_i}\vec{\sigma}_p - \frac{\partial \vec{h}_p}{\partial r_i}\cdot n_p\right] = \frac{2}{\hbar}\vec{h}_p\times\vec{\sigma}_p + I_{coll\ sp}$$

The second equation describles spin transport, integrate it over momentum space. And notice that interaction conserve spin, thus the $\int dp\, I_{sp\ coll} = 0 \Rightarrow$

$$\frac{\partial}{\partial t}\vec{\sigma}(r,t) + \frac{\partial}{\partial r_i}\vec{j}_i(r,t) = 0$$ (if there's no exteral magnetic field)

$$\vec{\sigma}(r,t) = 2\int\frac{d^3p}{(2\pi)^3}\,\vec{\sigma}(r,p,t)$$ (a factor 2 comes from trace of $I_{2\times 2}$ matrix)

$$\vec{j}_i(r,t) = 2\int\frac{d^3p}{(2\pi)^3}\left[\frac{\partial \mathcal{E}_p}{\partial p_i}\vec{\sigma}(r,p,t) + \frac{\partial \vec{h}_p}{\partial p_i} n_p(r,p,t)\right]$$

in Fermi liquid system, $\vec{\sigma}_p$, $\vec{h}_p$ are fluctuation effects, thus are linerly proportional to δn

$$\mathcal{E}_{p\sigma}(r,t) = \varepsilon_p + \int \frac{dp'}{(2\pi)^3} f_{p\sigma p'\sigma'} \delta n_{p'\sigma'}$$

$$\Rightarrow \vec{h}_p = \frac{\varepsilon_{p\uparrow} - \varepsilon_{p\downarrow}}{2} = \frac{1}{2}\int \frac{d^3p'}{(2\pi)^3} f_{\uparrow\uparrow}(pp')\delta n_{p'\uparrow} + f_{\uparrow\downarrow}(pp')\delta n_{p'\downarrow} - f_{\downarrow\uparrow}(pp')\delta n_{p'\uparrow} - f_{\downarrow\downarrow}(pp')\delta n_{p'\downarrow}$$

$$= \frac{1}{2}\int \frac{d^3p'}{(2\pi)^3} (f_{\uparrow\uparrow} - f_{\downarrow\uparrow})_{pp'} (\delta n_{p'\uparrow} - \delta n_{p'\downarrow}) = 2\int \frac{d^3p'}{(2\pi)^3} f^a_{pp'} \cdot \vec{\sigma}_{p'}$$

linearizing $\vec{j}_i \Rightarrow$ $\vec{j}_i = 2\int \frac{d^3p}{(2\pi)^3} \left[\frac{\partial \varepsilon_p^0}{\partial p_i} \vec{\sigma}(r,p,t) + \frac{\partial \vec{h}_p}{\partial p_i} n_p(r,p,t) \right]$

$$= 2\int \frac{d^3p}{(2\pi)^3} v_{p_i}\left[\vec{\sigma}(r,p,t) - \vec{h}_p \cdot \frac{\partial n_p}{\partial \varepsilon}\right]$$

$$\int \frac{d^3p}{(2\pi)^3} \underbrace{\left(-\frac{\partial n_p}{\partial \varepsilon_p}\right)}_{v_{p_i}} \vec{h}_p = 2\int \frac{d^3p}{(2\pi)^3} \underbrace{\left(-\frac{\partial n_p}{\partial \varepsilon_p}\right)}_{v_{p_i}} \int \frac{d^3p'}{(2\pi)^3} f^a_{pp'} \cdot \vec{\sigma}_{p'}$$

$$= N(0) \int \frac{d^3p'}{(2\pi)^3} \left[\int \frac{d\Omega_p}{4\pi} \sum_l f^a_l P_l(\cos\theta_{pp'}) v_{p_i}\right] \cdot \vec{\sigma}_{p'}$$

in doing $\int \frac{d\Omega_p}{4\pi} \sum_l f^a_l P_l(\cos\theta_{pp'}) v_{p_i}$ we set $\hat{z}$ of $\hat{p}$ along direction of $\hat{p}'$

$\Rightarrow \vec{v}_p = v_F \cos\theta\, \hat{z} + \vec{v}_\perp$, the $\vec{v}_\perp$ averages to zero; only the longitudinal part survive $\Rightarrow \frac{1}{3} F^a_1 v_{p'}$

$$\Rightarrow \int \frac{d^3p}{(2\pi)^3} v_{p_i} \left(-\frac{\partial n_p}{\partial \varepsilon_p}\right) \vec{h}_p = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{3} F_1^a v_{p'_i} \cdot \vec{\sigma}_{p'}(r,t)$$

$$\Rightarrow \vec{j}_i = 2\int \frac{d^3p}{(2\pi)^3} v_{p_i} \left[1+\frac{1}{3}F_1^a\right] \vec{\sigma}_p(r,t) = 2\int \frac{d^3p}{(2\pi)^3} \left(1+\frac{1}{3}F_1^a\right)\frac{P_i}{m^*} \vec{\sigma}_p(r,t)$$

$\Rightarrow$ we can define spin-effective mass $\dfrac{1}{m_s^*} = \dfrac{1+\frac{1}{3}F_1^a}{m^*} \Rightarrow$

$$\boxed{\frac{m_s^*}{m} = \frac{1+\frac{1}{3}F_1^s}{1+\frac{1}{3}F_1^a}}$$

3. ① Sound velocity in fluid (hydrodynamic sound / the first sound)

$$\frac{\partial}{\partial t} n + \nabla(n\vec{v}) = 0 \quad \text{and} \quad m n \frac{\partial \vec{v}}{\partial t} = -\nabla P$$

linearize $n = \bar{n} + \delta n \Rightarrow$

$$\left.\begin{aligned} &\frac{\partial}{\partial t}\delta n + \bar{n}\,\nabla\vec{V} = 0 \\ &m\bar{n}\,\frac{\partial \vec{v}}{\partial t} = -\nabla p(n+\delta n) \end{aligned}\right\} \Rightarrow \frac{\partial^2}{\partial t^2}\delta n = -\bar{n}\,\nabla \frac{\partial}{\partial t}\vec{V} = \frac{1}{m}\nabla^2 p(n+\delta n)$$

Pressure as function of density

$$\nabla^2 p(n+\delta n) = \frac{\partial P}{\partial n}\nabla^2 \delta n$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\delta n = \frac{1}{m}\frac{\partial P}{\partial n}\nabla^2\delta n \Rightarrow C_1^2 = \frac{1}{m}\frac{\partial P}{\partial n}.$$

$$\because \quad nV = \text{const} \quad V\,dn + n\,dV = 0 \Rightarrow \frac{\partial P}{\partial n} = -\frac{V}{n}\frac{\partial P}{\partial V}$$

$$\Rightarrow C_1^2 = \left[\frac{1}{mn}\,\frac{-V\,dp}{dV}\right] = \left[\frac{1}{P\chi}\right] \qquad \chi = \frac{1}{n^2}\frac{\partial n}{\partial \mu}$$

$$C_1^2 = \left[\frac{n^2}{mn}\frac{\partial \mu}{\partial n}\right] = \left[\frac{n}{m}\frac{\partial \mu}{\partial n}\right]$$

For idea Fermi gas $\mu \propto n^{\frac{2}{3}}$ $\quad \partial\mu \propto \frac{2}{3} n^{-1/3}\,\partial n \Rightarrow \frac{\partial \mu}{\partial n} = \frac{2}{3}\frac{\mu}{n}$

$$C_1^2 = \left[\frac{\mu}{m}\cdot\frac{2}{3}\right] = \left[\frac{1}{3}v_F^2\right] \qquad C_1 = \frac{v_F}{\sqrt{3}} \text{ for ideal Fermi gas}$$

For interacting Fermi liquid $\quad \frac{\partial \mu}{\partial n} = \left(\frac{m^*}{m}\right)^{-1}(1+F_0^s)\left(\frac{\partial \mu}{\partial n}\right)_0 = \frac{1+F_0^s}{1+\frac{1}{3}F_1^s}\left(\frac{\partial \mu}{\partial n}\right)_0$

$$\Rightarrow C_{1,FL}^2 = \frac{1+F_0^s}{1+\frac{1}{3}F_1^s}\,\frac{1}{3}\left[\frac{\hbar k_F}{m}\right]^2$$

3.b) From the Boltzman equation

$$\frac{\partial}{\partial t} n_p(r,t) + \nabla_p \varepsilon_p(r,t)\, \nabla_r n_p(r,t) - \nabla_r \varepsilon_p(r,t)\, \nabla_p n_p(r,t) = I_{coll}$$

linearing the equation

$$\varepsilon_p(r,t) = \varepsilon_0(p) + \frac{1}{V}\sum_{p'} f^s(p,p')\, \delta n_{p'}(rt)$$

$$n(prt) = n_0(p) + \delta n_p(rt)$$

$\nabla_r n_p(r,t)$ and $\nabla_r \varepsilon_p(r,t)$ are already at linear order of δn_p, thus we keep $\nabla_p \varepsilon_p(r,t)$ and $\nabla_p n_p(r,t)$ as zero-th order

$$\frac{\partial}{\partial t}\delta n_p(r,t) + \vec{v}_p \cdot \nabla_r \delta n_p(r,t) - \nabla_p n_p^0 \left(\int \frac{dp'}{(2\pi)^3} f^s_{pp'} \nabla_r \delta n_{p'}(rt)\right) = I_{coll}$$

$$\frac{\partial}{\partial t}\delta n_p(r,t) + \vec{v}_p \cdot \nabla_r \left[\delta n_p(r,t) - \frac{\partial n_p^0}{\partial \varepsilon} \int \frac{dp'}{(2\pi)^3} f^s_{pp'} \nabla_r \delta n_{p'}(rt)\right] = I_{coll}.$$

do Fourier transform $\delta n_p(r,t) = \sum_q \delta n_p \, e^{i(qr-\omega t)}$

we have

$$(-i\omega + i\vec{v}_p\cdot\vec{q})\,\delta n_p - \frac{\partial n^0}{\partial \varepsilon}\, \vec{v}_p \cdot i\vec{q} \left(\int \frac{dp'}{(2\pi)^3} f^s_{pp'}\, \delta n_{p'}\right) = I_{coll}$$

$$(\omega - vq\cos\theta_p)\,\delta n_p + \frac{\partial n^0}{\partial \varepsilon}\, vq\cos\theta_p \int \frac{dp'}{(2\pi)^3} f^s_{pp'}\, \delta n_{p'} = \frac{\delta n_p}{\tau}\, i$$

relaxation time approx

at $\omega\tau \gg 1$, the collision integral can be neglected.

we have

$$\boxed{(s - \cos\theta_p)\,\delta n_p - \left(-\frac{\partial n^0}{\partial \varepsilon_p}\right)\cos\theta_p \int \frac{d^3p'}{(2\pi)^3} f^s_{pp'}\, \delta n_{p'} = 0}$$

try the distribution $\delta n_p = -\frac{\partial n_p^0}{\partial \varepsilon_p} \nu_{\hat{p}}$. where $\nu_{\hat{p}}$ only depends on the direction of $\hat{p}$

$$\Rightarrow (s-\cos\theta_p)\,\nu_{\hat{p}} - \cos\theta_p \int \frac{d^3p'}{(2\pi)^3} \left(-\frac{\partial n_{p'}^0}{\partial \varepsilon_{p'}}\right) f_{pp'}\, \nu_{\hat{p}'} = 0$$

To solve this equation, expand $\nu_{\hat{p}} = \sum_{l0} Y_{l0}(\hat{p})\, u_l$. set the direction of $\vec{q}$ along $\hat{z}$-axis

$$\Rightarrow \sum_l (s-\cos\theta_p)\, Y_{l0}(\hat{p})\, u_l - \cos\theta_p \int \frac{d^3p'}{(2\pi)^3} \left(-\frac{\partial n_p^0}{\partial \varepsilon_p}\right) \left(\sum_{l'm'} f_{l'}^s \frac{4\pi}{2l'+1} Y_{l'm'}(\hat{p})\, Y_{l'm'}^*(\hat{p}')\right) \sum_l Y_{l0}(\hat{p}')\, u_l = 0$$

$$\sum_l Y_{l0}(\hat{p})\, u_l - \int \frac{d^3p'}{(2\pi)^3} \left(-\frac{\partial n_{p'}^0}{\partial \varepsilon_{p'}}\right) \frac{\cos\theta_p}{s-\cos\theta_p} \sum_{\substack{l'm'\\ l}} f_{l'}^s \frac{4\pi}{2l'+1} \left(Y_{l'm'}^*(\hat{p}')\, Y_{l0}(\hat{p}')\right) \underbrace{Y_{l'm'}(\hat{p})}_{u_l=0}$$

$$\int \frac{d^3p'}{(2\pi)^3} \left(-\frac{\partial n_{p'}^0}{\partial \varepsilon_{p'}}\right) \longrightarrow N(0) \int \frac{d\Omega_{p'}}{4\pi} \quad \Rightarrow \text{integrate over } p'$$

$$\sum_l Y_{l0}(\hat{p})\, u_l - \int \frac{d\Omega_{p'}}{4\pi} \frac{\cos\theta_p}{s-\cos\theta_p} \sum_{\substack{l'm'\\ l}} F_{l'}^s \frac{4\pi}{2l'+1} \left(Y_{l'm'}^*(\hat{p}')\, Y_{l0}(\hat{p}')\right) Y_{l'm'}(\hat{p})\, u_l = 0$$

$$\boxed{\sum_l Y_{l0}(\hat{p})\, u_l - \frac{\cos\theta_p}{s-\cos\theta_p} \sum_l F_l^s \frac{1}{2l+1} Y_{l0}(\hat{p})\, u_l = 0}$$

$$\int d\Omega_p\, Y_{l'm'}^*(\hat{p}) \sum_l Y_{l0}(\hat{p})\, u_l - \int d\Omega_p \frac{\cos\theta_p}{s-\cos\theta_p} Y_{l'm'}^*(\hat{p}) \sum_l \frac{F_l^s}{2l+1} Y_{l0}(\hat{p})\, u_l = 0$$

$$\Rightarrow u_{l'} - \sum_l \int d\Omega_p\, Y_{l'0}^*(\hat{p}) \frac{F_l^s}{2l+1} Y_{l0}(\hat{p})\, u_l \frac{\cos\theta_p}{s-\cos\theta_p} = 0$$

exchange l and l', and use $Y_{l0}(\hat{p}) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta_p) \Rightarrow$

$$u_l - \sum_{l'} \int \frac{d\Omega_p}{4\pi} \frac{\sqrt{2l+1}}{\sqrt{2l'+1}} F^s_{l'} P_l(\cos\theta_p) P_{l'}(\cos\theta_p) \frac{\cos\theta_p}{s-\cos\theta_p} u_{l'} = 0$$

$$\frac{u_l}{\sqrt{2l+1}} - \sum_{l'} \int \frac{d\Omega_p}{4\pi} P_l(\cos\theta_p) P_{l'}(\cos\theta_p) \frac{\cos\theta_p}{s-\cos\theta_p} F^s_{l'} \frac{u_{l'}}{\sqrt{2l'+1}} = 0$$

keep $l=0 \Rightarrow$ $\quad u_0 = \int \frac{d\Omega_p}{4\pi} \frac{\cos\theta_p}{s-\cos\theta_p} F^s_0\, u_0$

i.e. $-\frac{1}{F^s_0} = \int \frac{d\Omega_p}{4\pi} \frac{-\cos\theta_p}{s-\cos\theta_p} = 1 + \frac{s}{2} \ln\left|\frac{s-1}{s+1}\right| + i\frac{\pi}{2}s\,\Theta(1-s)$

as we showed in class. at $F^s_0 > 0$, there exist solution at $s>1$, which is beyond the particle-hole continuum.

at $s \to 1 \Rightarrow 1 + \frac{s}{2}\ln\left|\frac{s-1}{s+1}\right| \simeq$

or $F^s_0 \to 0^+$

$$1 + \frac{1}{2}\ln\frac{s-1}{2} = -\frac{1}{F^s_0}$$

$$\Rightarrow \ln\frac{s-1}{2} \approx -\frac{2}{F^s_0} \Rightarrow s-1 \approx 2e^{-\frac{2}{F^s_0}} \quad \text{i.e. } s = 1 + 2e^{-\frac{2}{F^s_0}}$$

at $F^s_0 \gg 1$ $\quad 1 + \frac{s}{2}\ln\left|\frac{s-1}{s+1}\right| \sim -\frac{1}{3s^2}$

$\Rightarrow \quad s = \sqrt{F_0/3}$ $\qquad$ s is the sound velocity.

③

$$\Omega_{00} = \int \frac{d\Omega_p}{4\pi} \frac{-\cos\theta_p}{s-\cos\theta_p} = 1 + \frac{s}{2}\ln\left|\frac{s-1}{s+1}\right|$$

by

$$\Omega_{10} = \Omega_{01} = \int \frac{d\Omega_p}{4\pi} \frac{-\cos^2\theta_p}{s-\cos\theta_p} = \int \frac{d\Omega_p}{4\pi}\left(1-\frac{s}{s-\cos\theta_p}\right)\cos\theta_p = s\,\Omega_{00}$$

$$\Omega_{11} = \int \frac{d\Omega_p}{4\pi} \frac{-\cos^3\theta_p}{s-\cos\theta_p} = \int \frac{d\Omega_p}{4\pi}\left(1-\frac{s}{s-\cos\theta_p}\right)(\cos^2\theta_p) = \frac{1}{3} + s\int \frac{d\Omega_p}{4\pi}\frac{-\cos^2\theta_p}{s-\cos\theta_p}$$

$$= \frac{1}{3} + s^2\,\Omega_{00}$$

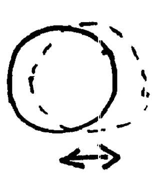
Clearly if only $F_{l=0} \neq 0 \Rightarrow$

$$\frac{u_l}{\sqrt{2l+1}} + \Omega_{l0} F_0^s \frac{u_{l'=0}}{\sqrt{1}} = 0, \quad \text{and} \quad \Omega_{00} F_0^s = -1$$

$$\Rightarrow u_l = \sqrt{2l+1}\,\frac{\Omega_{l0}}{\Omega_{00}}\,u_0 \quad \text{for } (l \geq 1)$$

because for the zero sound dispersion $s > 1 \Rightarrow u_1 = \sqrt{3}\, s\, u_0$

$\Rightarrow$ p-wave distortion is stronger than s-wave $\Rightarrow$



an osillation back-and forth of Fermi surface

Next, we assume both F_0^s and F_1^s are nonzero

$$-u_0 = \Omega_{00} F_0^s u_0 + s\Omega_{00} F_1^s \frac{u_1}{\sqrt{3}}$$

$$-\frac{u_1}{\sqrt{3}} = s\Omega_{00} F_0^s u_0 + (s^2\Omega_{00} + 1/3) F_1^s \frac{u_1}{\sqrt{3}}$$

$$\Rightarrow \begin{vmatrix} \Omega_{00} F_0^s + 1, & s\Omega_{00} F_1^s \\ s\Omega_{00} F_0^s, & (s^2\Omega_{00} + 1/3) F_1^s + 1 \end{vmatrix} = 0$$

$$(\Omega_{00} F_0^s + 1)\left[s^2\Omega_{00} F_1^s + \frac{1}{3} F_1^s + 1\right] - s^2\Omega_{00}^2 F_1^s F_0^s = 0$$

$$\Omega_{00} \frac{1}{3} F_0^s F_1^s + \Omega_{00} F_0^s + s^2\Omega_{00} F_1^s + 1 + \frac{F_1^s}{3} = 0$$

$$\Rightarrow -\Omega_{00} = \frac{1 + F_1^s/3}{F_0^s + s^2 F_1^s + \frac{1}{3} F_0^s F_1^s} = \frac{1}{2} s \ln\left(\left|\frac{s+1}{s-1}\right|\right) - 1$$

↑ zero sound with correction of F_1^s