Lecture 11.5 Landau Fermi liquid (V)

§ More on Luttinger theorem — Luttinger & Ward

Another expression: $$\frac{N}{V} = 2i\int\left[\frac{\partial}{\partial\omega}\ln G(p) - G(p)\frac{\partial}{\partial\omega}\Sigma(p)\right]e^{i\omega\eta}\frac{d^4p}{(2\pi)^4}$$

Proof:

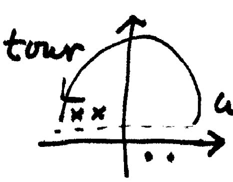
$$\frac{\partial}{\partial\omega}\ln G(p) - G(p)\frac{\partial}{\partial\omega}\Sigma(p) = G^{-1}(p)\frac{\partial}{\partial\omega}G(p) - G\frac{\partial}{\partial\omega}\Sigma = (\omega-\xi-\Sigma)\frac{\partial}{\partial\omega}G$$

$$-G\frac{\partial}{\partial\omega}\Sigma = \omega\frac{\partial}{\partial\omega}G - \xi\frac{\partial}{\partial\omega}G - \frac{\partial}{\partial\omega}(\Sigma G) = \frac{\partial}{\partial\omega}(\omega G) - \frac{\partial}{\partial\omega}(\Sigma G) - \frac{\partial}{\partial\omega}(\xi G) - G$$

$$= \frac{\partial}{\partial\omega}[(\omega-\xi-\Sigma)G] - G \Rightarrow \boxed{\frac{N}{V} = -2i\int G\, e^{i\omega\eta}\frac{d^4p}{(2\pi)^4}}$$

— The second term of the integral can be proved to be zero. Let us postpone its proof for a while. $$\frac{N}{V} = 2i\int\frac{\partial}{\partial\omega}\ln G(p)\, e^{i\omega\eta}\frac{d^4p}{(2\pi)^4}$$

Contour



G is not analytic. $G_R = \begin{cases} G(\epsilon) & \epsilon>0 \\ G^*(\epsilon) & \epsilon<0 \end{cases}$, G_R is analytic for ω in the upper plan

So

$$\frac{N}{V} = 2i\left[\int_0^{\infty}\frac{d\omega}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial}{\partial\omega}\ln G_R(p) + \int_{-\infty}^{0}\frac{d\omega}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial}{\partial\omega}\ln G(p)\right]$$

$$= 2i\left[\underbrace{\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial}{\partial\omega}\ln G_R(p)}_{=0} + \int_{-\infty}^{0}\frac{d\omega}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial}{\partial\omega}\ln\frac{G(\vec{p},\omega)}{G^*(\vec{p},\omega)}\right]$$

$$= \frac{2i}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\ln\frac{G(p,\omega)}{G^*(p,\omega)}\Bigg|_{-\infty}^{0} = -\frac{2}{\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}[\varphi(0)-\varphi(-\infty)],$$

where φ denotes the phase of G. We need to consider the variation of φ from $\omega = -\infty$ to $\omega = 0$. From the Lehmann Rep. we have

$$G(\omega) = e^{\beta\Omega} \sum_{n,m} \langle n|\psi|m\rangle\langle m|\psi^{\dagger}|n\rangle \left\{ \frac{e^{-\beta E_n}}{\omega + E_n - E_m + i\eta} + \frac{e^{-\beta E_m}}{\omega + E_n - E_m - i\eta} \right\}$$

$$\xrightarrow{T=0K} = \sum_m \frac{\langle 0|\psi|m\rangle\langle m|\psi^{\dagger}|0\rangle}{\omega - E_m + i\eta} + \sum_n \frac{\langle 0|\psi^{\dagger}|n\rangle\langle n|\psi|0\rangle}{\omega + E_n - i\eta}$$

$$= \int_0^{+\infty} dE \; \frac{A(E)}{\omega - E + i\eta} + \frac{B(E)}{\omega + E - i\eta} ,$$

$\boxed{A, B \text{ are positive}}$

where
$$A(E) = \sum_m \langle 0|\psi|m\rangle\langle m|\psi^{\dagger}|0\rangle \, \delta(E - E_m)$$
$$B(E) = \sum_n \langle 0|\psi^{\dagger}|n\rangle\langle n|\psi|0\rangle \, \delta(E - E_n) .$$

As $\omega \to \infty \Rightarrow G(\omega) \to \frac{1}{\omega} \int_0^{+\infty} (A(E) + B(E))\, dE$

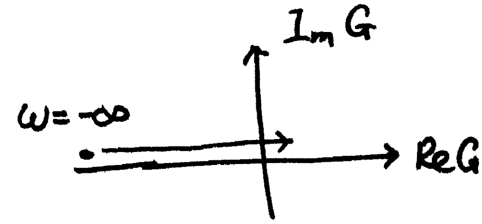
$$= \sum_{m,n} \frac{1}{\omega} \{\langle 0|\psi|m\rangle\langle m|\psi^{\dagger}|0\rangle + \langle 0|\psi^{\dagger}|n\rangle\langle n|\psi|0\rangle\} = \frac{1}{\omega}\langle 0|\psi\psi^{\dagger} + \psi^{\dagger}\psi|0\rangle$$

$$= \frac{1}{\omega} \quad \text{i.e.} \quad \boxed{G(\omega) \to \frac{1}{\omega}, \quad \text{as } \omega \to \infty} .$$

$$\mathrm{Re}\, G(\omega) = P\int_0^{+\infty} dE \; \frac{A(E)}{\omega - E} + \frac{B(E)}{\omega + E}$$

$$\mathrm{Im}\, G(\omega) = \begin{cases} -\pi A(\omega) & \text{for } \omega > 0 \\ \pi B(-\omega) & \text{for } \omega < 0 \end{cases} .$$

As $\omega \to -\infty$, $\mathrm{Re}\, G(\omega) \to -\frac{1}{\omega} < 0$. $\mathrm{Im}\, G(\omega)$ at $\omega < 0$, describe the hole excitation below the Fermi surface. If we assume that the hole-excitation is bounded from below, then $\mathrm{Im}\, G(\omega) > 0$ decays faster towards 0. $\Rightarrow$ $\varphi(\omega \to -\infty) = \pi$.



We set $\mathrm{Im}\, G(\omega) = 0$ at $\omega = 0$, because all the Z_n, E_m in the Lenhmann Rep > 0. The φ of $G(\omega=0)$ is determined by the real part $\mathrm{Re}\, G(\omega=0)$. If $\mathrm{Re}\, G(\omega=0) < 0 \Rightarrow \varphi = \pi$,

$\mathrm{Re}\, G(\omega=0) > 0 \Rightarrow \varphi = 0$.

$$\Rightarrow \boxed{\frac{N}{V} = 2 \int_{\mathrm{Re}\, G(p,\,\omega=0) > 0} \frac{d^3p}{(2\pi)^3}}$$

The region $G(p, \omega=0)$ is bound by a surface of either zero or diveregence.

Vanishing of $G(p, \omega=0)$ corresponds to $\Sigma \to \infty$, which corresponds to superconductivity

$$G = \frac{u_k^2}{\omega - \sqrt{\varepsilon_k^2 + \Delta^2}} + \frac{v_k^2}{\omega + \sqrt{\varepsilon_k^2 + \Delta^2}}.$$

$$\Rightarrow G(0, k) = \frac{-u_k^2 + v_k^2}{E_k},$$

(we do not consider this possibility here).

On the other hand, $G(p, \omega=0) = \dfrac{Z}{\omega - \xi_p + i\eta} + \cdots \to +\infty$

corresponds to location of $\xi_p = 0$, i.e. <u>the location of F S.</u>

$$G(p, \omega=0) \to -\frac{Z}{\xi_p} \to +\infty \quad \xi_p < 0$$

$$-\infty \quad \text{for } \xi_p > 0.$$

This proof does not assume the isotropy of F S. Interaction may deform the shape but not its volume!