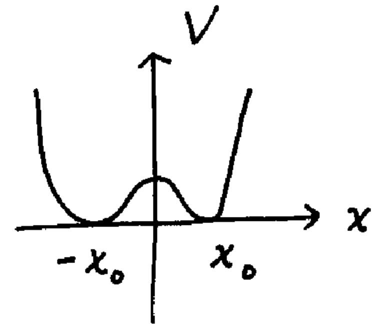
Lect 3 : More applications of path integral

Tunneling through a barrier



$$H = \frac{p^2}{2m} + V(x)$$

we use imaginary time path integral

$$\langle x_f | e^{-HT/\hbar} | x_i \rangle = N\int [Dx(t)]\, e^{-S/\hbar} = N\int [Dx(t)] \exp - \int_{-T/2}^{T/2} \frac{d\tau}{\hbar} \left(\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x) \right)$$

Let us search for classic path with the following boundary condition

$$x(\tfrac{T}{2}) = x_f, \quad x(-\tfrac{T}{2}) = x_i$$

$\Rightarrow$ the classic path $\bar{x}(t)$: $\dfrac{\delta S}{\delta \bar{x}} = -m \dfrac{d^2 \bar{x}}{d\tau^2} + V'(\bar{x}) = 0$

Let us do a small fluctuation $x(t) = \bar{x}(t) + \delta \bar{x}(t)$, $\boxed{\delta \bar{x}(t) = 0 \text{ at } T/2, -T/2}$

$$S = \int \left(\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + V(x) \right) dt = \int \frac{m}{2} \frac{d\bar{x}}{dt} + V(\bar{x})\, dt + \int_{-T/2}^{T/2} \frac{m}{2}\left(\frac{d\,\delta\bar{x}}{dt}\right)^2 + \frac{V''}{2}(\bar{x}(t)) (\delta(x(t)))^2$$

$$= S(\bar{x}(t)) + \int_{-T/2}^{T/2} d\tau\, \frac{m}{2}\, \delta x(t) \left(-\frac{d^2}{d\tau^2} + V''(\bar{x}(t)) \right] \delta x(t)$$

we can solve the spectrum

$$X_n(-\tfrac{T}{2}) = X_n(\tfrac{T}{2}) = 0$$

$$\left\{ -m \frac{d^2}{d\tau^2} + V''(\bar{x}(t)) \right\} X_n(t) = \lambda_n X_n(t)$$

$$\Rightarrow \langle x_f | e^{-H/\hbar \cdot T} | x_i \rangle \approx N e^{-S(\bar{x}(t))/\hbar} \prod_n \lambda_n^{-1/2}$$

$$= N e^{-S(\bar{x}(t))/\hbar} \left(\det[-\partial_\tau^2 + V''(\bar{x}(t))) \right)^{-1/2}$$

Lemma: how to calculate the determinant?

$[-m\partial_t^2 + W(t)]\,\psi = \lambda\psi$, where ψ satisfies the boundary condition $\psi(-T/2)=0$, $\partial_t\psi(-T/2)=1$.

$(-T/2 \le t \le T/2)$

we know if $\lambda=\lambda_n$, then we have $\psi(\frac{T}{2})=0$.

Let $W_1(t)$ and $W_2(t)$ be two functions of t, and $\psi_{1,2,\lambda}(t)$ be the corresponding solutions for the above boundary condition, we have

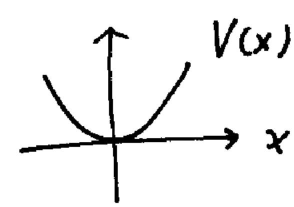
$$\det\left[\frac{-m\partial_t^2 + W_1(t)-\lambda}{-m\partial_t^2 + W_2(t)-\lambda}\right] = \frac{\psi_{1,\lambda}(\frac{T}{2})}{\psi_{2,\lambda}(T/2)}$$

Proof: both sides are semi-analytical functions of λ,

as $\lambda \to \lambda_{\substack{1,\\2} n}$, $\psi_{\substack{1,\\2}\lambda_n}(\frac{T}{2}) \to 0$ and $\det(-m\partial_t^2 + W_{1,2}(t)-\lambda) \to 0$.

Thus both sides have same pattern of zeros and poles. Again, as $|\lambda|\to+\infty$ both sides $\to 1$, thus the LHS and RHS must equal.

Let us apply this Lemma for the simplest case of harmonic ~~oscilattor~~ oscilator.



and we calculate $\langle 0|\, e^{-HT/\hbar}\,|0\rangle = N\cdot\det[-m\partial_t^2 + m\omega^2]^{-1/2}$

$$\det\left[\frac{\frac{m}{\hbar}(-\partial_\tau^2+\omega^2)}{\frac{m}{\hbar}(-\partial_\tau^2)}\right] = \frac{\psi_{\omega,\lambda=0}(\frac{T}{2})}{\psi_{\omega=0,\lambda=0}(\frac{T}{2})}$$

For $\omega=0,\ \lambda=0 \Rightarrow \psi_{\omega=0,\lambda=0}(t) = t+T/2$

$\omega=\omega,\ \lambda=0 \Rightarrow (\partial_\tau^2-\omega^2)\psi(t)=0 \Rightarrow \psi_{\omega,\lambda=0}(t) = \dfrac{\sinh(\omega(t+T/2))}{\omega}$

$$\Rightarrow \det\left[\frac{\frac{m}{\hbar}(-\partial_\tau^2+\omega^2)}{\frac{m}{\hbar}(-\partial_\tau^2)}\right] = \left.\frac{\sinh(\omega(t+T/2))}{\omega(t+T/2)}\right|_{t=T/2} = \frac{\sinh\omega T}{\omega T}$$

we need evaluate $\det\left[-\frac{m}{\hbar}\partial_\tau^2\right]$, which is the determinat for free space.

$$\langle 0|e^{-HT/\hbar}|0\rangle = N\det\left[\frac{m}{\hbar}\left(-\frac{\partial}{\partial t^2}\right)\right] = \int_{-\infty}^{+\infty}\frac{dp}{2\pi\hbar}\langle 0|p\rangle\langle p|0\rangle e^{-\frac{p^2T}{2m\hbar}}$$

$$= \frac{1}{2\pi\hbar}\left(\frac{\pi}{\frac{T}{2m\hbar}}\right)^{1/2} = \left(\frac{m}{2\pi\hbar T}\right)^{1/2}$$

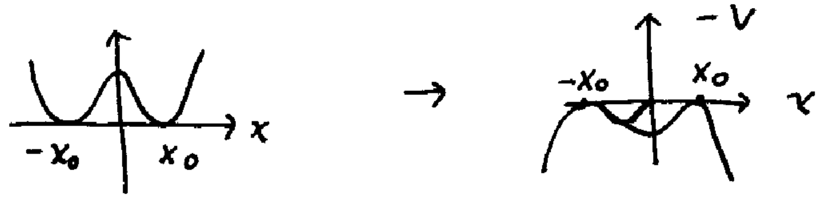
$$\Rightarrow \langle 0|e^{-\left(\frac{p^2}{2m}+\frac{m}{2}\omega^2x^2\right)T/\hbar}|0\rangle = \left(\frac{m}{2\pi\hbar T}\right)^{1/2}\left(\frac{\sinh\omega T}{\omega T}\right)^{-1/2} = \left(\frac{2\pi\hbar\sinh\omega T}{m\omega}\right)^{-1/2}$$

$$\xrightarrow[T\to\infty]{} \frac{m\omega}{\pi\hbar}e^{-\omega T/2}$$

Compare with $\langle 0|e^{-HT/\hbar}|0\rangle \xrightarrow{T\to+\infty} |\langle 0|0\rangle|_{n=0}^2 e^{-E_0T/\hbar} = \left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\right]^2 e^{-\omega T/2}$

which is correct!

now let us come back to the double well problem $V(x) = \frac{mg}{4}(x^2 - x_0^2)^2$

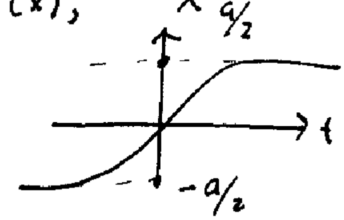


at $x = \pm x_0$
the local oscillation
frequency $\omega_0 \approx g x_0^2$

let us calculate

$$\langle a | e^{-HT/\hbar} | -a \rangle = N \int D x(\tau) \exp\left[- \int_{-T/2}^{T/2} dt \; \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x) \right]$$

The classical path $\rightarrow$ motion in the potential of $-V(x)$,



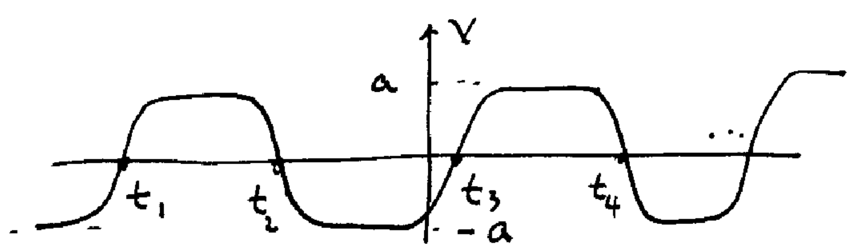
$$\frac{dx}{d\tau} = \sqrt{\frac{2V(x)}{m}} \quad \Rightarrow \int dt = \int \frac{dx}{\sqrt{2V(x)/m}} \quad \Rightarrow$$

$$t = -T/2 + \int_{-a}^{x} dx' \left(\frac{2V(x')}{m} \right)^{-1/2}, \qquad x(-T/2) = -a$$

$$S_0 = \int_{-T/2}^{T/2} \frac{d\tau}{\hbar} \left(\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x) \right) = \int_{-T/2}^{T/2} \frac{d\tau}{\hbar} \; \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 = \frac{m}{\hbar} \int_{-a}^{a} dx \left(\frac{dx}{dt}\right)$$

$$= \frac{m}{\hbar} \int_{-a}^{a} dx \sqrt{\frac{2mV(x)}{\hbar}}$$

at large $T \to +\infty$, other classical paths include. widely seperated instanton / anti-instantons



The leading order contribution $S = n S_0$

if $t \neq t_1, \cdots t_n \cdots$, particles are mainly around $x = \pm \frac{X_0}{2}$, where

$V''(x) = m\omega_0^2$, which gives the contribution $\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} e^{-\omega T/2}$ as before

Next. let us integrate out all the possible locations of centers

$$\int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{\cancel{T/2}\, t_1} dt_2 \cdots \int_{-T/2}^{\cancel{T/2}\, t_{n-1}} dt_n = \frac{T^n}{n!}$$

$$\rightarrow \langle a | e^{-H/\hbar \cdot T} | -a \rangle = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \sum_{n \in odd} \frac{(K e^{-S_0/\hbar} T)^n}{n!}$$

similarly $\langle -a | e^{-HT/\hbar} | -a \rangle = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \sum_{n \in even} \frac{(K e^{-S_0/\hbar} T)^n}{n!}$

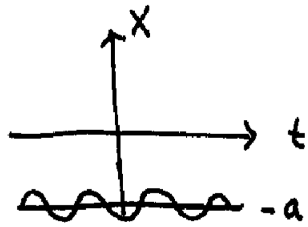
$$\Rightarrow \langle \pm a | e^{-H/\hbar \cdot T} | -a \rangle = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \begin{cases} \cosh(K e^{-S_0/\hbar}) \\ \sinh(K e^{-S_0/\hbar}) \end{cases}$$

$\rightarrow E_{1,2} = \frac{\hbar\omega_0}{2} \mp \hbar K e^{-S_0/\hbar}$, where K is a constant due to appearence of instanton/anti-instanton

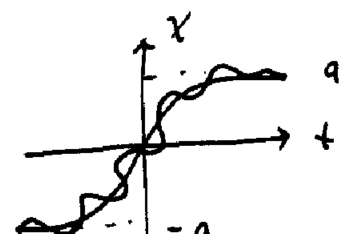
$$\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \rightarrow \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} K^n.$$

§ Evaluation of K,

Let us compare the configuration of zero instanton / one instanton



and



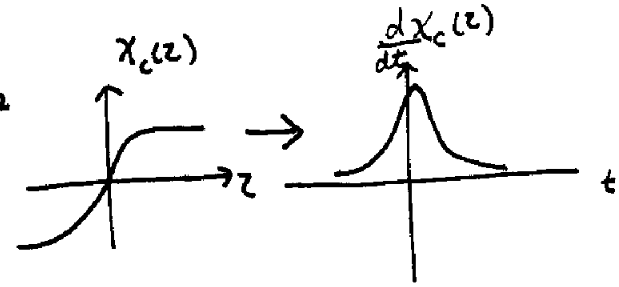
$$K e^{-S_0/\hbar} = \int_{x_c} D\delta x \, e^{-S/\hbar} \Big/ \int_{x = x_0} D\delta x \, e^{-S/\hbar} \qquad (X_0 = -a)$$

near $x = x_c(\tau) \Rightarrow \quad S = S_0 + \int d\tau \; \delta x \left(-\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V''(x_c(\tau))\right)\delta x$

$x = x_0 \quad \Rightarrow \quad S = \int d\tau \; \delta x \left(-\frac{m}{2}\frac{d^2}{d\tau^2} + V''(x_0)\right)\delta x$

$$\Rightarrow K = \frac{\int_{x_c} D\delta x \; e^{-\int d\tau \, \delta x \left(-\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V''(x_c(\tau))\right)\delta x}}{\int_{x=x_0} D\delta x \; e^{-\int d\tau \, \delta x \left(-\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V''(x_0)\right)\delta x}}$$

$$= \left(\frac{\det\left(-m\frac{d^2}{d\tau^2} + V''(x_c)\right)}{\det\left(-m\frac{d^2}{d\tau^2} + V''(x_0)\right)}\right)^{-1/2}$$



But $-m\frac{d^2}{d\tau^2} + V''(x_c)$ has an zero energy mode :

because the saddle point equation : $-m\frac{d^2 x_c(\tau)}{d\tau^2} + V'(x_c(\tau)) = 0$

$$\Rightarrow -m\frac{d^2}{d\tau^2}\left[\frac{d}{d\tau}x_c(\tau)\right] + V''(x_c(\tau))\left[\frac{d}{d\tau}x_c(\tau)\right] = 0 .$$

This zero mode corresponds to the time translational symmety of the instanton solution. This contibution has already been included

$$K \cdot T = \frac{\int_{x_c} D\delta x \; e^{-S}}{\int_{x=x_0} D\delta x \; e^{-S}}$$

$\leftarrow$ the zero frequecy part contribute to "$\bar{\bar{T}}$".

In order to give a careful analysis, we discreteze the integral as

$$\int_{x_c} D\delta x\, e^{-S} = A_N \int \prod_{i=1}^{N} d\delta x_i \; \exp\left[-\frac{\Delta\tau}{2} \sum_{i=1}^{N} \delta x_i \left(-m\frac{d^2}{d\tau^2} + V''(x_{c,i})\right)\delta x_i\right]$$

$$= A_N \cdot \left(\sqrt{\frac{\pi 2}{\Delta\tau}}\right)^N \Big/ \left\{\mathrm{Det}\left[-m\frac{d^2}{d\tau^2} + V''(x_c)\right]\right\}^{1/2}$$

let us expand δx_i, interms of eigenstates of $-m\frac{d^2}{d\tau^2} + V''(x_c)$

$$\delta x(\tau_i) = c_1 \varphi_1(\tau_i) + \sum_{n=2}^{N} c_n \varphi_n(\tau_i),$$ where $\varphi_j(\tau_i)$ is normalized

to $\quad \Delta\tau \sum_i \varphi_j(\tau_i)\varphi_{j'}(\tau_i) = \delta_{ij} \Rightarrow$

$$\int_{x_c} D\delta x\, e^{-S} = A_N \int \prod_{j=1}^{N} dc_j \sqrt{\Delta\tau} \cdot \exp\left[-\sum_{j=1}^{N} \frac{\lambda_j}{2} c_j^2\right]$$

$$= A_N \int dc_1 \sqrt{\Delta\tau} \int \prod_{j=2}^{N} dc_j \sqrt{\Delta\tau}\, \exp\left[-\sum_{j=2}^{N} \frac{\lambda_j}{2} c_j^2\right]$$

$$= A_N \int dc_1 \sqrt{\Delta\tau} \cdot \left(\frac{\sqrt{2\pi}}{\sqrt{\Delta\tau}}\right)^{N-1} \mathrm{Det}'\left[-m\frac{d^2}{d\tau^2} + V''(x_c)\right]^{-1/2} \leftarrow \boxed{\text{Zero mode excluded}}$$

$$\Rightarrow \int_{-T/2}^{T/2} d\tau \; K = \lim_{N\to+\infty} \frac{Z_N(x_c)}{Z_N(x_0)} = \frac{\int dc_1 \sqrt{\Delta\tau} \left(\sqrt{\frac{2\pi}{\Delta\tau}}\right)^{N-1} \mathrm{Det}'\left[-m\frac{d^2}{d\tau^2} + V''(x_c)\right]^{-1/2}}{\left(\sqrt{\frac{2\pi}{\Delta\tau}}\right)^{N} \mathrm{Det}\left[-m\frac{d^2}{d\tau^2} + V''(x_0)\right]^{-1/2}}$$

$$= \int \frac{dc_1}{\sqrt{2\pi}} \cdot \left\{\frac{\mathrm{Det}'\left[-m\frac{d^2}{d\tau^2} + V''(x_c)\right]}{\mathrm{Det}\left[-m\frac{d^2}{d\tau^2} + V''(x_0)\right]}\right\}^{-1/2}$$

next. we need to figure out the relation between $\int d\tau$ and $\int dc_1$

The normalized $\varphi_1(\tau)$ should be : $\varphi_1(\tau) = \dfrac{\dot{x}_c(\tau)}{\sqrt{S_0/m}}$, because

$$S_0 = \int d\tau \ \frac{1}{2} m (\dot{x}_c)^2 + V(x_c) = \Delta\tau \sum_{i=1}^{N} m \left(\frac{\Delta x_c}{\Delta \tau}\right)^2$$

$$\Rightarrow \Delta\tau \sum_{i=1}^{N} \left(\frac{\Delta x_c}{\Delta\tau} \Big/ \sqrt{\frac{S_0}{m}} \right)^2 = 1 \quad \Rightarrow \quad \varphi_1(\tau) = \dot{x}_c \Big/ \sqrt{\frac{S_0}{m}}$$

$$\Rightarrow \delta x_{(\tau)} = \left(C_1 \Big/ \sqrt{\frac{S_0}{m}} \right) \dot{x}_c(\tau) \quad \text{correspond to a shift} \quad \Delta\tau = \frac{C_1}{\sqrt{\frac{S_0}{m}}}$$

~~Since~~ $\Rightarrow$ interval for C_1 $\Rightarrow \sqrt{\frac{S_0}{m}} \left(-\frac{T}{2}, \ \frac{T}{2}\right)$

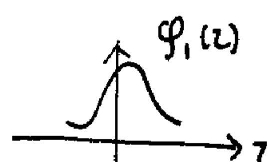
$$\Rightarrow \quad K \cdot T = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{S_0}{m}} \cdot T \lim_{N \to +\infty} \left\{ \frac{\mathrm{Det}' \left[-m \frac{d^2}{d\tau^2} + V''(x_c) \right]}{\mathrm{Det} \left[-m \frac{d^2}{d\tau^2} + V''(x_0) \right]} \right\}^{-1/2}$$

$$\Rightarrow \quad K = \sqrt{\frac{S_0}{2\pi}} \lim_{N \to +\infty} \left\{ \frac{\mathrm{Det}' \left[-\frac{d^2}{d\tau^2} + \frac{1}{m} V''(x_c) \right]}{\mathrm{Det} \left[-\frac{d^2}{d\tau^2} + \frac{1}{m} V''(x_0) \right)} \right\}^{-1/2}$$

~~Det~~ operator $\left[-\frac{d^2}{d\tau^2} + \frac{1}{m} V''(x_0) \right]$'s eigenvalues are all positive.

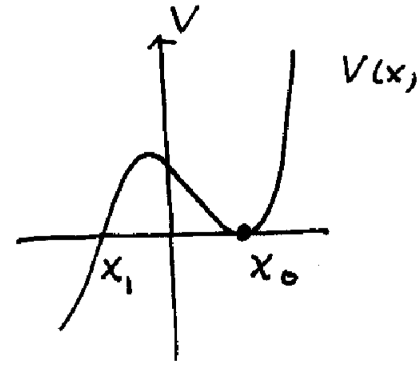
how about operator $\left[-\frac{d^2}{d\tau^2} + \frac{1}{m} V''(x_c) \right]$? as we know it contains one

zero mode, with $\varphi_1(\tau)$



This one does not have nodes, which must the eigenstate with the lowest energy. As a result, all other eigenvalues are positive, and K is real.

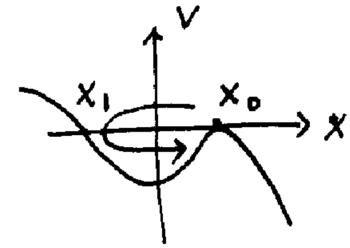
§ Fate of a meta-stable state



let us consider the potential $V(x)$, and put particle at x_0, we calculate $\langle x_0 | e^{-HT} | x_0 \rangle$

$$\langle x_0 | e^{-HT} | x_0 \rangle = N \int D x[\tau] \; \exp - \int_{-T/2}^{T/2} \frac{d\tau}{\hbar} \left[\frac{m}{2} \left(\frac{dx}{d\tau} \right)^2 + V(x) \right].$$

The classical path or $m \dfrac{d^2 x(\tau)}{d\tau^2} = V'(x) \rightarrow$



Again we have

$$\langle x_0 | e^{-HT} | x_0 \rangle = e^{-T\omega_0/2} \sum_n \int_{-T/2}^{T/2} d\tau_1 \int_{\tau_1}^{T/2} d\tau_2 \cdots \int^{T/2} d\tau_n \; (K e^{-S_0})^n$$

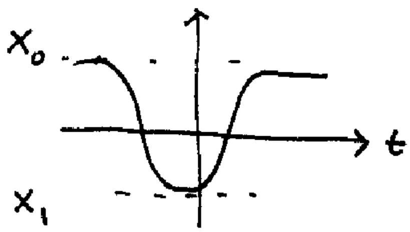
$$= e^{(TKe^{-S_0})} \, e^{-T\omega_0/2},$$

How ever, in this case, the coefficient K is not real, but an imaginary number. If in the real time, representation, we have

$$\langle x_0 | e^{-iHT} | x_0 \rangle \rightarrow e^{-iT\frac{\omega_0}{2} - \underbrace{T|K| e^{-S_0}}}$$

the decay probility.

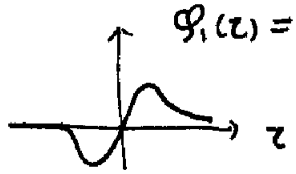
The different from the double well problem is that, we do not have an instanton solution, but a bounce solution.



particle can't be stable at x_1, and must come back!

The its time derivative, i.e. the zero mode of $-\frac{d^2}{d\tau^2} + V''(x_c)$ behaves like

$$\varphi_1(\tau) = \frac{d}{d\tau} x_c(\tau)$$



, this one node.

Thus there must be another eigenstate with lower energy, (i.e. negative).

As a result, $\left\{ \mathrm{Det}' \left(-\frac{d^2}{d\tau^2} + \frac{1}{m} V''(x_c) \right) \right\}^{-1/2}$ becomes imaginary, which leads to decay process.