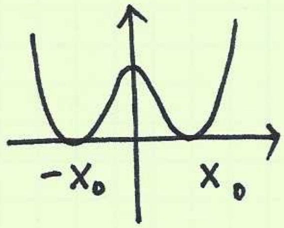
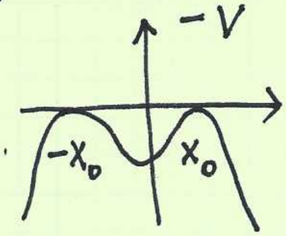
Instantons: double well problem

§ Consider the potential $V(x) = \frac{mg}{4}(x^2 - x_0^2)^2$



→

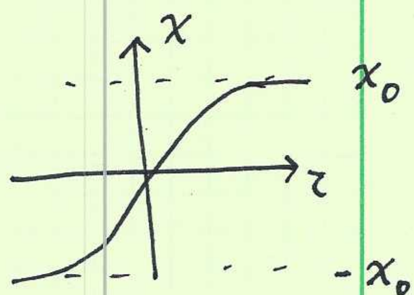


at $x = \pm x_0$, the local frequency $\omega_0 \simeq g x_0^2$,

Now we calculate $\langle x_0 | e^{-HT/\hbar} | -x_0 \rangle = A \int Dx(\tau)\, e^{-\int_{-T/2}^{T/2} d\tau \left[\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)\right]}$

① figure out the classic path → motion in the potential of $-V(x)$

$$\frac{dx}{d\tau} = \sqrt{\frac{2V(x)}{m}} \Rightarrow \int d\tau = \int \frac{dx}{\sqrt{2V(x)/m}}$$

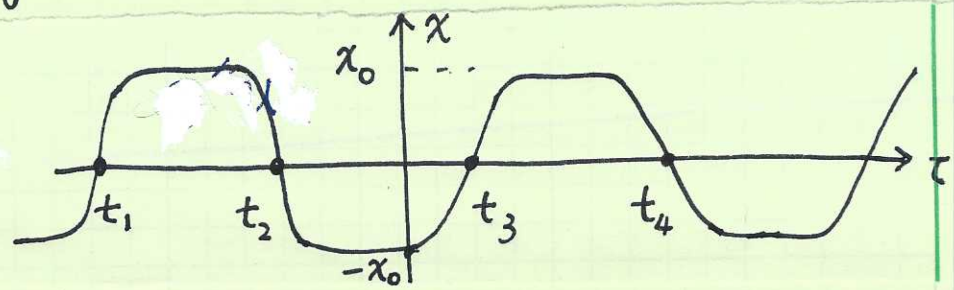


$$\Rightarrow \begin{cases} \tau = -T/2 + \int_{-x_0}^{x} dx' \left(\frac{2V(x)}{m}\right)^{-1/2} \\ x(-T/2) = -x_0 \end{cases}$$

classic action $S_0 = \int_{-T/2}^{T/2} \frac{d\tau}{\hbar} \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x) = \int_{-T/2}^{T/2} \frac{d\tau}{\hbar}\, m \left(\frac{dx}{d\tau}\right)^2$

$$= \int_{-x_0}^{x_0} dx \left(\frac{dx}{d\tau}\right) \frac{m}{\hbar} = \frac{m}{\hbar} \int_{-x_0}^{x_0} dx \sqrt{\frac{2mV(x)}{\hbar}}$$

At large $T \to +\infty$, we can have other classic paths:



a configuration with n-instanton / anti-instanton

The leading order contribution $S = nS_0 \leftarrow$ contribution around $t \simeq t_1, t_2, \ldots t_4$.

turning point contribution

At other time, i.e $t \neq t_1\, t_2, \cdots$, the particle is mainly at $\pm x_0$ where $V''(x) = m\omega_0^2$. This gives the contribution $\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} e^{-\omega T/2}$

Now let's integrate out all the possible location of centers.

$$\int_{-T/2}^{T/2} d\tau_1 \int_{-T/2}^{\tau_1} d\tau_2 \cdots \int_{-T/2}^{\tau_{n-1}} d\tau_n = \frac{T^n}{n!} \Rightarrow$$

$$\langle x_0 | e^{-HT/\hbar} | -x_0 \rangle = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \sum_{n\in odd} \frac{\left(K e^{-S_0/\hbar} T\right)^n}{n!}$$

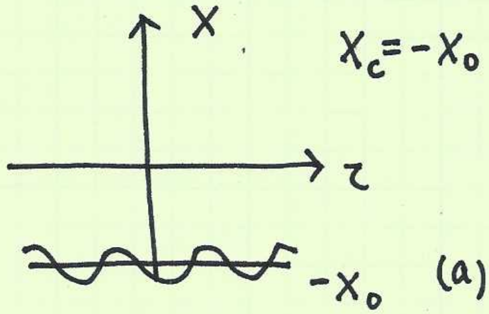
Similarly $\langle -x_0 | e^{-HT/\hbar} | -x_0 \rangle = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \sum_{n\in even} \frac{\left(K e^{-S_0/\hbar} T\right)^n}{n!}$

$$\Rightarrow \langle \pm x_0 | e^{-HT/\hbar} | -x_0 \rangle = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \begin{cases} \cosh\left(KTe^{-S_0/\hbar}\right) \\ \sinh\left(KTe^{-S_0/\hbar}\right) \end{cases}.$$

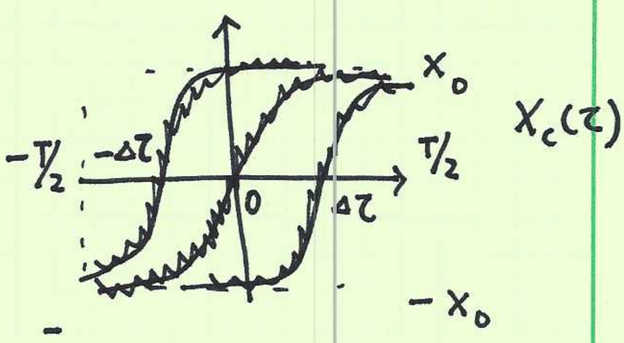
Extract the time dependent part $\Rightarrow E_{1,2} = \frac{\hbar\omega_0}{2} \mp \hbar K e^{-S_0/\hbar}$.

§ Evaluation of K (K carry unit of frequency)

Compare the sector of zero instanton, and one instanton



(a)



instanton can occur at any time between $-T/2$ and $T/2$

define
$$K T e^{-S_0/\hbar} = \frac{\int_{x_c} Dx \; e^{-S/\hbar}}{\int_{x \simeq -x_0} Dx \; e^{-S/\hbar}}$$
← around (b)

← config around (a)

then LHS and RHS both dimensionless.

For RHS, we only need to choose an instanton configuration located at $\tau = 0$. The other instantons are not independent, but can be viewed as zero mode fluctuations, i.e if

$X_c(\tau)$ is a classic solution, so does $X_c(\tau + \Delta\tau)$

$$\Rightarrow \quad X_c(\tau + \Delta\tau) = X_c(\tau) + \Delta\tau \frac{d}{d\tau} X_c(\tau)$$

Since $X_c(\tau+\Delta\tau)$ and $X_c(\tau)$ give the same minimal action $\Rightarrow$

$\frac{d}{d\tau}X_c(\tau)$ is a zero mode. We can prove explicitly

the classic solution satisfies $-m\frac{d^2X_c(\tau)}{d\tau^2}+V'(X_c(\tau))=0$

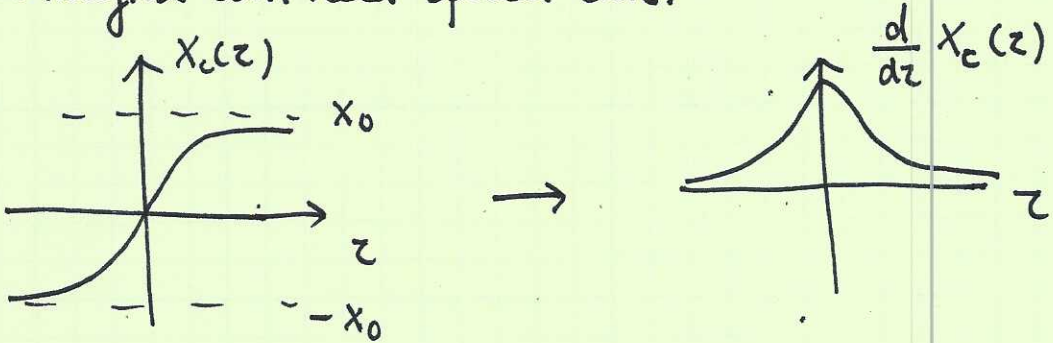
$$\Rightarrow \boxed{-m\frac{d^2}{d\tau^2}\left(\frac{d}{d\tau}X_c(\tau)\right)+V''(X_c(\tau))\left(\frac{d}{d\tau}X_c(\tau)\right)=0}$$

Now near $X=X_c(\tau)$, $S=S_0+\int_{-T/2}^{T/2}d\tau\,\delta x\left[-\frac{m}{2}\frac{d^2}{d\tau^2}+\frac{V''}{2}(X_c(\tau))\right]\delta X$

$$\Rightarrow \boxed{\int_{x_c} DX\,e^{-S/\hbar}=e^{-S_0/\hbar}\int_{x_c} D\delta X\;e^{-\frac{1}{\hbar}\int d\tau\,\delta x\left(\underline{-\frac{m}{2}\frac{d^2}{d\tau^2}+\frac{V''}{2}(X_c(\tau))}\right)\delta X}}$$

Since $\frac{d}{d\tau}X_c(\tau)$ is a zero mode of $-\frac{m}{2}\frac{d^2}{d\tau^2}+V''(X_c(\tau))$, the

~~f~~ path integral will need special care.



Let us discretize the functional integral to have a better understanding.

$$\int_{x_c} D\delta X\;e^{-\frac{1}{\hbar}\int d\tau\,\delta x\left(-\frac{m}{2}\frac{d^2}{d\tau^2}+V''\right)\delta X}=A\int\prod_{i=1}^{N}d\,\delta X_i\;\exp\Big[-\frac{\Delta\tau}{2\hbar}\sum_{\substack{i=1\\j}}^{N}\delta X_i\left(-m(\Delta\tau)^{-2}\Delta+V''\right)\delta X_j$$

$$= A\left(\sqrt{\frac{2\pi}{\Delta\tau}}\right)^N \Big/ \left\{\mathrm{Det}\left[-m(\Delta\tau)^{-2}\Delta + V''\right]\right\}^{1/2}$$ (Δ is the discretized laplacian)

Now let's expand δX_i in terms of orth normal basis of eigenstates of $-m\frac{d^2}{d\tau^2} + V''$

$$\delta X_i = C_1\varphi_1(\tau_i) + \sum_{n=2}^{N} C_n \varphi_n(\tau_i)$$ and $\varphi_n(\tau_i)$ is normalized

as $\sum_i \Delta\tau \varphi_j(\tau_i)\varphi_{j'}(\tau_i) = \delta_{ij}$ $\Rightarrow$ $\det\left[(\Delta\tau)^{1/2}\varphi_j(\tau_i)\right] = 1$

orth-normal

or $\det[\varphi_j(\tau_i)] = (\Delta\tau)^{-\frac{N}{2}}$

The Jaccbian change from $\prod_{i=1}^{N} dX_i \to \prod_{i=1}^{N} dC_n$

$$\prod_{i=1}^{N} dX_i = \prod_{i=1}^{N} dC_i \left|\frac{\partial X_i}{\partial C_j}\right| = \prod_{i=1}^{N} dC_i \det[\varphi_j(\tau_i)]$$

$$= \prod_{i=1}^{N}\left[dC_i (\Delta\tau)^{-1/2}\right]$$

$$\Rightarrow \int_{x_c} D\delta x\, e^{-S} = \underbrace{A}_{e^{-S_0/\hbar}} \int \prod_{i=1}^{N}\left[dC_i(\Delta\tau)^{-1/2}\right] \exp\left[-\sum_{j=1}^{N}\frac{\Delta\tau}{2}\lambda_j C_j^2\right]$$

$$= \underbrace{A}_{e^{-S_0/\hbar}} \int dC_1(\Delta\tau)^{-1/2} \int \prod_{i=2}^{N} dC_i(\Delta\tau)^{-1/2} \exp\left[-\sum_{j=2}^{N}\frac{\Delta\tau}{2}\lambda_j C_j^2\right]$$

$$= \underbrace{A}_{e^{-S_0/\hbar}} \int dC_1(\Delta\tau)^{-1/2}\left(\frac{2\pi}{\Delta\tau}\right)^{\frac{N-1}{2}} \mathrm{Det}'\left[-m\frac{d^2}{d\tau^2} + V''(x_c)\right]$$ zero mode excluded

$$\Rightarrow K T e^{-S_0/\hbar} = \frac{A e^{-S_0/\hbar} (\Delta\tau)^{-\frac{N}{2}} \left(\frac{2\pi}{\hbar}\right)^{\frac{N-1}{2}} \left[\mathrm{Det}'\left[-m\frac{d^2}{d\tau^2} + V''(x_c)\right]\right]^{-1/2} \int dC_1}{A (\Delta\tau)^{-N/2} (2\pi\hbar)^{\frac{N}{2}} \left\{\mathrm{Det}\left[-m\frac{d^2}{d\tau^2} + V''(-x_0)\right]\right\}^{-1/2}}$$

$$K T = \int \frac{dC_1}{\sqrt{2\pi\hbar}} \left\{ \frac{\mathrm{Det}'\left[-m\frac{d^2}{d\tau^2} + V''(x_c)\right]}{\mathrm{Det}\left[-m\frac{d^2}{d\tau^2} + V''(-x_0)\right]} \right\}^{-1/2}$$

now we need to figure out the relatin $\int dC_1$ and $\int_{-T/2}^{T/2} d\tau = T$.

Let's normalize $\varphi_1(\tau) \propto \dot{x}_c(\tau)$ ← (zero mode)

$$S_0 = \int d\tau \; \frac{1}{2} m \dot{x}_c^2 + V(x_c) = \int d\tau \; m \dot{x}_c^2 = m \Delta\tau \sum_{i=1}^{N} \left(\frac{\Delta x_c}{\Delta\tau}\right)^2$$

$$\Rightarrow \Delta\tau \sum_{i=1}^{N} \left(\frac{\dot{x}_c(\tau_i)}{\sqrt{S_0/m}}\right)^2 = 1 \quad \Rightarrow \varphi_1(\tau) = \frac{\dot{x}_c(\tau)}{\sqrt{S_0/m}}$$

For $\delta x(\tau) = \frac{C_1}{\sqrt{S_0/m}} \dot{x}_c(\tau)$, → correspond a shift

$x_c(\tau) \to x_c(\tau + \Delta\tau)$ with

$$\Delta\tau = \frac{C_1}{\sqrt{S_0/m}}$$

Since the interval for $\Delta\tau \in \left[-\frac{T}{2}, \frac{T}{2}\right] \Rightarrow C_1$'s interval

$$-\sqrt{S_0/m}\,\frac{T}{2} \le C_1 \le \sqrt{S_0/m}\,\frac{T}{2} \quad \Rightarrow \int dC_1 = \sqrt{\frac{S_0}{m}} \int d\tau = \sqrt{\frac{S_0}{m}}\, T$$

$$\Rightarrow \quad KT = \sqrt{\frac{S_0}{2\pi m\hbar}}\, T \lim_{N\to\infty} \left\{ \frac{\mathrm{Det}'\left[-m\frac{d^2}{d\tau^2} + V''(x_c)\right]}{\mathrm{Det}\left[-m\frac{d^2}{d\tau^2} + V''(-x_0)\right]} \right\}^{-1/2}$$

$$\boxed{K = \sqrt{\frac{S_0}{2\pi\hbar}} \lim_{N\to\infty} \left\{ \frac{\mathrm{Det}'\left[-\frac{d^2}{d\tau^2} + \frac{1}{m}V''(x_c)\right]}{\mathrm{Det}\left[-\frac{d^2}{d\tau^2} + \frac{1}{m}V''(-x_0)\right]} \right\}^{-1/2}}$$

please note the absorbtion of "m".

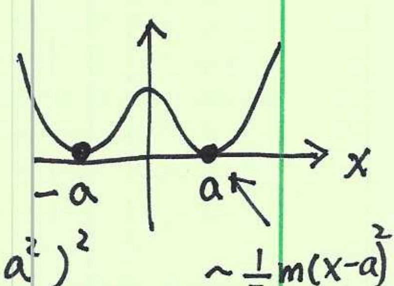
Check unit: eigenvalues of Det has unit T^{-2}, Det$'$ has one less eigenvalues ~~that~~ then Det, $\Rightarrow [K] = (T^{-2})^{-1\times(-1/2)} = T^{-1}$

K carries frequency's unit, correct!

We have shown that eigenvalues of $-\frac{d^2}{d\tau^2} + \frac{1}{m}V''(-x_0)$ are all positive definite. How about $-\frac{d^2}{d\tau^2} + \frac{1}{m}V''(x_c)$?

Since its zero mode $\varphi_1(\tau)$ is nodeless, it must be the mode with lowest eigenvalue, so all other modes' eigenvalue must be positive

$$\Rightarrow \boxed{K \text{ is real.}}$$



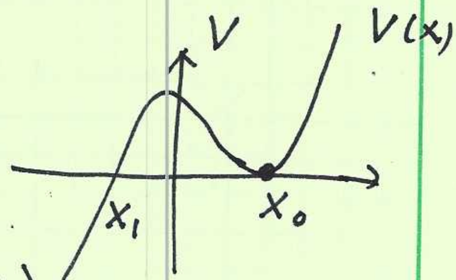
Consider a concrete potential $V(x) = \dfrac{m\omega^2(x^2-a^2)^2}{8a^2}$.

it can be calculated in J. Zinn-Justin (QFT and critical phenomena 1993)

$$\left\{ \begin{array}{l} \dfrac{\det'\left[-\frac{d^2}{d\tau^2} + V''(x_{cl})\right]}{\det\left[-\frac{d^2}{d\tau^2} + \omega^2\right]} = \dfrac{1}{12}\omega^{-2} \\ S_0 = \dfrac{2}{3} m\omega a^2 \end{array} \right.$$

$$\Rightarrow \boxed{K = \sqrt{\frac{2m\omega a^2}{3\pi\hbar \cdot 2}}\sqrt{12}\,\omega = 2\omega\sqrt{\frac{m\omega a^2}{\hbar}}}$$

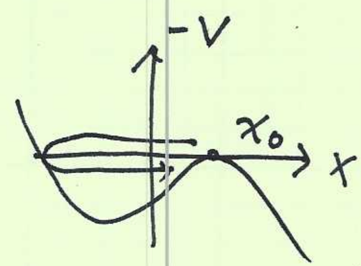
§ Decay of a false vaccum



Consider the potential $V(x)$, at put particle at x_0. We calculate $\langle x_0 | e^{-HT} | x_0 \rangle$

$$\langle x_0 | e^{-HT} | x_0 \rangle = A \int D[x(\tau)] \exp\left[-\int_{-T/2}^{T/2} \frac{d\tau}{\hbar} \left(\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)\right)\right]$$

The classic path $m \dfrac{d^2 x(\tau)}{d\tau^2} = V'(x)$



Again we have

$$\langle x_0 | e^{-HT} | x_0 \rangle = e^{-T\omega/2} \sum_n \int_{-T/2}^{T/2} d\tau_1 \int_{\tau_1}^{T/2} d\tau_2 \cdots \int_{\tau_{n-1}}^{T/2} d\tau_n \left(K e^{-S_0}\right)^n$$

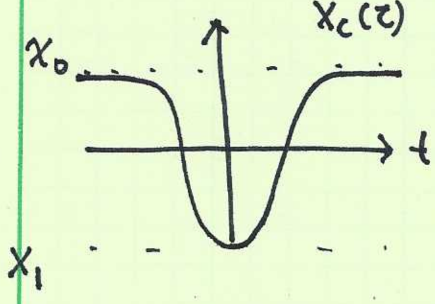
$$= \exp\left[K T e^{-S_0}\right] e^{-T\omega_0/2}$$

However, the "K" here is not real, but imaginary. If we go back to real time, we have $T \to iT$,

$$\langle x_0 | e^{-iHT} | x_0 \rangle = e^{-iT\frac{\omega_0}{2} \underbrace{-T|K|e^{-S_0}}}$$

the decay probability.

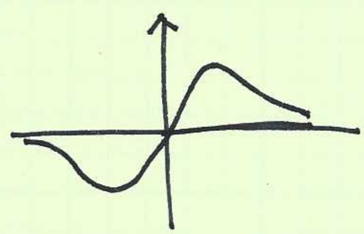
The difference from the double well solution: we don't have an instanton, but a "bounce" solution for e^{-S_0}



the particle cannot stop at x_1 and must come back.

The zero mode of $-\frac{d^2}{d\tau^2} + V''(x_c)$

behaves like:



$\varphi_1(\tau) = \frac{d}{d\tau} x_c(\tau)$

Since $\varphi_1(\tau)$ has one node, there must be an eigenmode with lower energy (i.e. negative). As a result, $\left[\mathrm{Det}\left[-\frac{d^2}{d\tau^2} + \frac{1}{m}V''(x_c)\right]\right]^{-1/2}$

becomes imaginary. $\rightarrow$ $K = |K|i$