Lect 7. Superfluidity at low dimensions.

— At low dimensions, strong quantum fluctuations can destroy long range superfluid order. Let us calculate the correlation function $\langle \varphi^\dagger(t\,x)\, \varphi(00) \rangle$. We assume $\varphi(xt)$ has a non-vanishing amplitude, but fluctuating phase.

$$\mathcal{L} = \frac{\chi}{2} \left((\partial_t \theta)^2 - v^2 (\partial_x \theta)^2 \right), \text{ where } \chi = \frac{1}{V_0}, \quad v = \sqrt{\frac{\rho_0 V_0}{m}}.$$

$$\langle \varphi^\dagger(t x)\, \varphi(00) \rangle = |\varphi_0|^2 \langle e^{-i\theta(tx)} e^{i\theta(00)} \rangle = \frac{\int D\theta\, e^{-i\theta(tx)} e^{i\theta(00)} e^{iS}}{\int D\theta\, e^{iS}},$$

Introducing source field J, and

— $$Z[J] = \int D\theta \; e^{iS + i\int dt\, d^dx\, J(x,t)\, \theta(t,x)}$$

$$\langle \varphi^\dagger(t,x)\, \varphi(00) \rangle = Z[-\delta(t-t_0, x-x_0) + \delta(t,x)] / Z[0],$$

$$\frac{Z[J(x,t)]}{Z[0]} = \frac{\int D\theta \; e^{\frac{i}{2}\int d^dx\, dt\; \theta(x_1 t_1)\, G_\theta^{-1}(x_1 t_1, x_2 t_2)\, \theta(x_2 t_2) + iJ(xt)\,\theta(x,t)}}{Z(0)}$$

$$= \int D\theta \; e^{\frac{i}{2}\int d^dx_1 dt_1 \int d^dx_2 dt_2 \left\{ [\theta(x_1 t_1) + \int dx_1' dt_1' \underbrace{G(x_1' t_1', x_1 t_1)]}_{J(x_1' t_1')} G^{-1}(x_1 t_1, x_2 t_2) \cdot \{\theta(x_2 t_2) + \int dx_2' dt_2'\, G(x_2 t_2, x_2' t_2)\, J(x_2' t_2')\} \right\}}$$

$$\cdot \exp\Big[-\frac{i}{2} \int d^dx_1 dt_1\, d^dx_2 dt_2\, d^dx_1' dt_1'\, d^dx_2' dt_2' \; J(x_1' t_1')\, G(x_1' t_1', x_1 t_1)$$

— $$G^{-1}(x_1 t_1, x_2 t_2)\, G(x_2 t_2, x_2' t_2')\, J(x_2' t_2')) / Z(0)$$

$$= \exp\Big[-\frac{i}{2} \int d^dx_1' dt_1'\, dx_2 dt_2'\; J(x_1' t_1')\, G(x_1' t_1', x_2' t_2')\, J(x_2' t_2') \Big]$$

extra (-) comes from partial integration

where G_θ is the inverse of $-\chi(+\partial_t^2 - v^2\partial_\theta^2)$. As we have shown before

$$\langle T[\theta(t x).\theta(0,0)\rangle = \frac{\int D\theta.\ \theta(t,x)\,\theta(00)\, e^{\frac{i}{2}\int dx_1 dt_1 dx_2 dt_2\, \theta(x_1 t_1)\, G^{-1}(x_1 t_1; x_2 t_2)\,\theta(x_2 t_2)}}{\int D\theta\ e^{\frac{i}{2}\int dx_1 dt_1 dx_2 t_2\, \theta(x_1 t_1)\, G^{-1}(x_1 t_1; x_2 t_1)\,\theta(x_2 t_2)}}$$

$$= (-i)\overset{-1}{G}{}^{+1}(t,x;00) = i\,G(tx;00)$$

$$\Rightarrow \langle e^{-i\theta(t_0 x_0)}\, e^{i\theta(00)}\rangle = e^{-i\frac{1}{2}\int dt_1 dx_1 \int dt_2 x_2 \left\{[-\delta(t_1-t_0,\, x_1-x_0) + \delta(t_1, x_1)] \times G(t_1-t_2,\, x_1-x_2)\,(-\delta(t_2-t_0,\, x_2-x_0) + \delta(t_2 x_2)]\right\}}$$

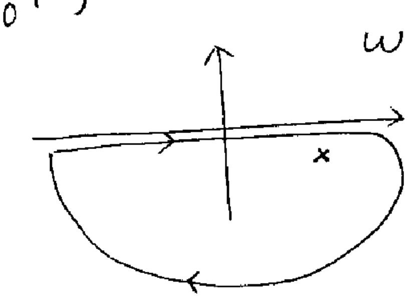
$$= e^{-\frac{i}{2}[2G(0,0) - G(t_0,x_0;00) - G(-t_0-x_0;00)]} = e^{i[G(tx) - G(00)]}$$

$$G(\omega,k) = \frac{\chi^{-1}}{\omega^2 - v^2k^2 + i0^+} = \frac{\chi^{-1}}{2vk}\left[\frac{1}{\omega - vk + i0^+} - \frac{1}{\omega + vk - i0^+}\right]$$

★ at 1D

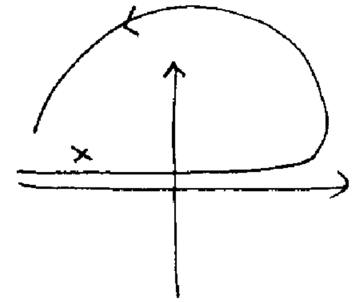
$$G(tx;00) = \int \frac{dk\,d\omega}{(2\pi)^2}\, G(\omega,k)\, e^{-i\omega t + ikx}$$

$$= \chi^{-1}\int \frac{dk\,d\omega}{(2\pi)^2}\, \frac{1}{2v|k|}\left[\frac{1}{\omega - vk + i0^+} - \frac{1}{\omega + vk - i0^+}\right] e^{-i\omega t + ikx}$$



if $t>0$

$$\chi^{-1}\int \frac{dk}{(2\pi)^2}\, \frac{-i\pi 2}{2v|k|}\, e^{-i(v|k| - i0^+)t + ikx}$$



if $t<0$

$$\chi^{-1}\int\frac{dk}{(2\pi)^2}\frac{-i\pi 2}{2v|k|}\, e^{i(v|k|t)+ikx}$$

$$\Rightarrow \boxed{G(tx;00) = \chi^{-1}\int\frac{dk}{(2\pi)^2}\frac{-i\pi}{v|k|}\, e^{i(-v|k||t|+kx)-0^+|t|}}$$

For finite system with length L, $k=\frac{2\pi}{L}\times$ integer, $\int\frac{dk}{2\pi}\to L^{-1}\sum_k$,

we get

$$G(tx;00) = \chi^{-1}L^{-1}\sum_k\frac{-i}{2v|k|}\, e^{i(-v|k||t|+kx)-0^+|t|}$$

$$=\frac{i}{4\pi v\chi}\left[\ln\left(1-e^{2\pi i\frac{-v|t|+x}{L}-0^+}\right)+\ln\left(1-e^{2\pi i\frac{-v|t|-x}{L}-0^+}\right)\right]$$

$$=\frac{i}{4\pi v\chi}\ln\left[\left(1-e^{2\pi i\frac{x-v|t|}{L}-0^+}\right)\left(1-e^{2\pi i\frac{-x-v|t|}{L}-0^+}\right)\right]$$

$$=\frac{i}{4\pi v\chi}\ln\left(-\left(\frac{2\pi i}{L}\right)^2(x-v|t|+i0^+)(x+v|t|-i0^+)\right)$$

$$=\frac{i}{4\pi v\chi}\ln\left(\frac{4\pi^2}{L^2}(x^2-v^2t^2+i0^+)\right).$$

$$\Rightarrow \langle e^{-i\theta(tx)}e^{i\theta(00)}\rangle = e^{iG(xt)}e^{-iG(00)} = e^{-iG(00)}\left(\frac{L^2}{4\pi^2(x^2-v^2t^2+i0^+)}\right)^{\frac{1}{4\pi v\chi}}$$

$$= e^{-iG(00)}\, e^{-i\frac{\pi}{4\pi v\chi}\Theta(v^2t^2-x^2)}\left(\frac{L^2}{4\pi^2|x^2-v^2t^2|}\right)^{\frac{1}{4\pi v\chi}}$$

The $G(00)$ is actually a short length scale cut off "ℓ"

$$\Rightarrow G(00) \to G_0(t=0, x=\ell) \Rightarrow \langle e^{-i\theta(t-x)} e^{i\theta(00)} \rangle = \left(\frac{\ell^2}{x^2 - v^2t^2 + i0^+} \right)^{\frac{1}{4\pi v \chi}}$$

point splitting

$$= e^{-i\frac{\pi}{4\pi v\chi}\Theta(v^2t^2 - x^2)} \left(\frac{\ell^2}{|x^2 - v^2t^2|} \right)^{\frac{1}{4\pi v \chi}} .$$

Thus we do not have true long range order, but a quasi-long range order with power law decay.

* at 2D:

We first calculate in the imaginary time representation

$$\mathcal{G}(x^i) = +T_\tau \langle \theta(x^i)\,\theta(0) \rangle , \quad i=1,2 \text{ for spatial coordinates}, \quad i=3 \text{ for imaginary time}$$

$$Z = \int D\theta \; e^{-\int d^3x \, \frac{\chi}{2}\left[v^2(\partial_x\theta)^2 + v^2(\partial_y\theta)^2 + (\partial_\tau\theta)^2\right]}$$

$$T_\tau \langle \theta(x^i)\,\theta(0) \rangle = \frac{\int D\theta \;\; \theta(x_1 x_2 \tau)\,\theta(000)\; e^{-\int d^3x_1 d^3x_2 \frac{\chi}{2}\theta(x_1^i)\, G^{-1}(x_1^i, x_2^i)\,\theta(x_2^i)}}{\int D\theta \;\; e^{-\int d^3x_1 d^3x_2 \frac{\chi}{2}\theta(x_1^i)\, G^{-1}(x_1^i, x_2^i)\,\theta(x_2^i)}}$$

$$= \quad , \chi^{-1} G(x^i; 0)$$

where $(-\partial_\tau^2 - v^2\partial_x^2)\, G(x^i; 0) = \delta^{(3)}(x)$, From electromagnetism we know

$$(-\partial_\tau^2 - v^2\partial_x^2)\left(\frac{1}{\sqrt{\tau^2 + \left(\frac{x}{v}\right)^2}} \right) = 4\pi\, \delta(\tau)\, \delta(x_{1/v})\, \delta\!\left(\frac{x_2}{v}\right)$$

$$\Rightarrow -(\partial_\tau^2 + v^2\partial_x^2)\left(\frac{1}{4\pi}\frac{1}{v\sqrt{x^2+(v\tau)^2}}\right) = \delta(\tau)\,\delta(x_1)\,\delta(x_2)$$

$$\Rightarrow \mathcal{G}(x^i) = T_\tau\langle\theta(x^i)\,\theta(0)\rangle = \frac{1}{4\pi\chi v}\frac{1}{\sqrt{x^2+(v\tau)^2}}$$

$$i\,G(t\,x_1,x_2) = \mathcal{G}(x_1 x_2;\, e^{i\pi/2}t) = \frac{1}{4\pi\chi v}\frac{1}{\sqrt{x^2-(vt)^2}},$$

at $|x|<|vt|$, how to decide the phase angle, we need define $\mathcal{G}(x_1 x_2;\, e^{i\theta}t)$ and let θ from $0\to\frac{\pi}{2}$, and we continuously $\underset{\text{as}}{\frown}$ from $\mathcal{G}\to iG$.

Thus
$$i\,G(t x_1 x_2) = \frac{1}{4\pi\chi v}\frac{1}{\sqrt{|x^2-(vt)^2|}}\,e^{-i\frac{\pi}{2}\,\Theta(v|t|-x)}.$$

$$\langle e^{-i\theta(tx)}\,e^{i\theta(00)}\rangle = e^{iG(tx)-iG(00)} \qquad \text{as } t;x\to+\infty,$$

we have $G(t.x)\to 0$; we also set a point-splitting for $G(00)$

$$\to G(0,\ell) = \frac{-i}{4\pi\chi v\ell}$$

$$\Rightarrow \langle e^{-i\theta(tx)}\,e^{i\theta(00)}\rangle \to e^{-\frac{1}{4\pi v\chi\ell}} = \langle e^{-i\theta(tx)}\rangle\langle e^{i\theta(00)}\rangle$$

Thus at 2+1 dimension. Quantum fluctuation does not destroy the long range order. but reduce the value of long range order.

§ Short-distance cut-off and upper critical dimension

We showed that quantum fluctuations will suppress the amplitude of order parameter, which depends on the short range length scale l. (cut off)
If $l \gg (v\chi)^{-1}$, then the correction will be small. How to define "l"? It should be no smaller than the "healing length" which describes the amplitude mode fluctuations: $\frac{(\partial_x \delta\rho)^2}{8m\rho_0} + \frac{V_0}{2}(\delta\rho)^2$

$$\Rightarrow \xi^2 = (4m\rho_0 V_0)^{-1}.$$

Below this length scale, we enter a high energy region characterized by fluctuation of amplitude mode, which cannot be described by XY-model.

Seating $l = \xi \Rightarrow$

$$e^{-\frac{1}{4\pi v\chi l}} = e^{-\frac{mV_0}{\sqrt{2}\pi}} = e^{-\sqrt{2}\frac{E_{int}}{E_{qua}}}.$$

(check dimension at 2+1 d, V_0 has the unit of $[E]^{-1}[L]^{-2}$, which is correct.

$$E_{int} = \frac{1}{2}V_0\rho_0 \qquad E_{qua} = \frac{N}{d^2 m} \sim \rho_0 m$$

the energy scale below which Boson statistic is important.

Let us do a scale transformation:

$$S = \int d^d x\, dt \left\{ \left[i\frac{1}{2}(\varphi^* \partial_t \varphi - \varphi \partial_t \varphi^*) \right] - \frac{1}{2m}\partial_x \varphi^* \partial_x \varphi - \frac{V_0}{2}(|\varphi|^2 - \rho_0)^2 \right.$$

let us rescale $t, x, \varphi, \cdot$

$$x = \xi \tilde{x}, \quad t = (V_0 \rho_0)^{-1} \tilde{t}, \quad \varphi = \sqrt{\rho_0}\, \tilde{\varphi}$$

$$\Rightarrow S = g^{-1} \int d^d\tilde{x}\, d\tilde{t}\, [i \tfrac{1}{2} (\tilde{\varphi}^* \partial_{\tilde{t}} \tilde{\varphi} - \tilde{\varphi} \partial_t \tilde{\varphi}^*] - z \partial_x \tilde{\varphi}^* \partial_x \tilde{\varphi} - \tfrac{1}{2}(|\tilde{\varphi}|^2 - 1)^2$$

where $g = (\rho_0 \xi^d)^{-1} = [\rho_0^{\,d-2} (4 m V_0)^d]^{1/2}$.

If g is small, the potential in $\int D^2\tilde{\varphi}\, e^{i \tilde{S}/g}$ is steep, and fluctuation are small. Otherwise, if g is large, fluctuations are large.

Thus as $d \geq 2$. as $\rho_0 \to 0$, we have weak coupling theory at quantum critical point.

as $d < 2$, as $\rho_0 \to 0$, we have to include strong quantum fluctuations.