Lecture 5 — Landau levels and quantum Hall effect

$$H_{2D} = \frac{(\vec{P} - \frac{e}{c}\vec{A})^2}{2m}, \quad \text{where} \quad \vec{A} = +B(y, 0),$$ $\boxed{\text{only depends on } y\text{, not on } x}$

Then $(\nabla \times \vec{A})_z = \partial_x A_y - \partial_y A_x = -B$; this gauge is first used by Landau thus is called Landau gauge. It's simpler than the symmetric gauge $\vec{A} = \frac{-B}{2}\hat{z} \times \vec{r}$, but it doesn't preserve rotation symmetry, explicitly.

$$H_{2D} = \frac{(P_x - \frac{e}{c}By)^2}{2m} + \frac{P_y^2}{2m} = \frac{P_y^2}{2m} + \frac{1}{2}m\omega_c^2\left(y - \frac{l_B^2 P_x}{\hbar}\right)^2$$

where $\omega_c = \frac{|eB|}{mc}$, $l_B = \sqrt{\frac{\hbar c}{|eB|}}$.

We solve the wavefunctions $\psi_{n,k_x}(\vec{r}) = f_n(y)\, e^{ik_x x}$

$$\Rightarrow \left[\frac{P_y^2}{2m} + \frac{1}{2}m\omega_c^2 (y - l_B^2 k_x)^2\right] f_n = E_n f_n$$

$\Rightarrow$ $E_n = (n + \frac{1}{2})\hbar\omega_c$, which is independent of k_x.

and $f_n(y) = \phi_n(y - y_0(k_x))$ which is a center-shifted harmonic oscillator wavefunction

$$\phi_n(x) = \left[\frac{1}{\sqrt{\pi}\, 2^n n!\, l_B}\right]^{1/2} H_n\left(\frac{x}{l_B}\right) e^{-\frac{x^2}{2l_B^2}},$$ where H_n is the

★ Hermite polynormial.

$H_0(x) = 1$, $H_1(x) = 2x$, $H_3(x) = 4x^2 - 2$, ...

a general formula for Hermite polynomial is

$$\boxed{H_n(x) = (-)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)}$$

it's generation function is $\boxed{e^{-s^2+2xs} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} s^n}$

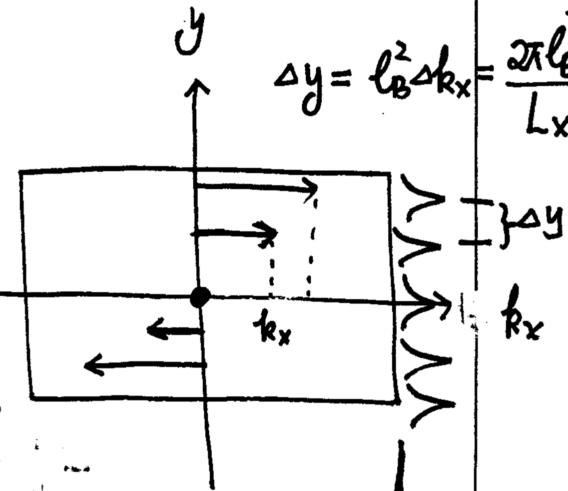
orthnormal condition

$$\boxed{\int_{-\infty}^{+\infty} H_m(x)\, H_n(x)\, e^{-x^2}\, dx = \sqrt{\pi}\, 2^n\, n!\, \delta_{nm}.}$$

You can look them up in any math-physics ~~text~~books.

§ spatial separation of chiral modes and non-commutative geometry.

$\Delta y = l_B^2 \Delta k_x = \frac{2\pi l_B^2}{L_x}$



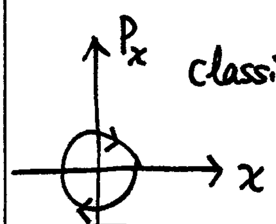
The system behaves like a set one-dimensional harmonic oscillators along the y-direction.

(Let us focus on the LLL with $n=0$, such that all the oscillators are in the viabrational ground states)

The interesting thing is that the centers of these harmonic modes are correlated with its momentum along the x-direction. For those with $k_x > 0$, their centers are shifted up, and for those with $k_y < 0$, their centers are shifted down.

In ~~other words, the y-axis plays the role of the momentum of~~ the x-axis. If we only keep the LLL states (this is justified in the case of the gap between LLs $\hbar\omega_c$ is much larger than all the other energy scales that we are interested, i.e. in the limit of $\hbar\omega_c \to +\infty$. This process is called LLL projection).

$$\boxed{[x, y]_{LLL} = [x, l_B^2 P_x/\hbar] = i l_B^2} \leftarrow \text{non-commutative geometry}$$



classic physics $\xrightarrow[\text{QM mechanics}]{[x, P_x] = i\hbar}$ the classic orbits are quantized $\oint p\,dx = nh$

$\boxed{\text{Classic orbits in phase space has chirality!}}$

in usual QM $[x, y] = 0$,

after LLL projection, $[x, y]_{LLL} = i l_B^2$.

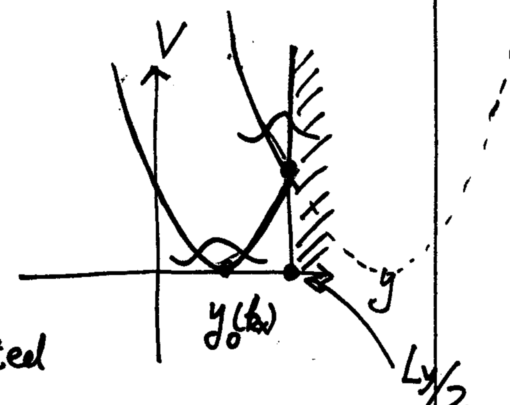
Thus the 2D LL in the LLL projection, the (xy)-plane behaves as the phase space of (x, k_x).

☆ Edge spectra:

The effective potential for the state with mementum k_x, is

$$V_{k_x} = \frac{1}{2} m \omega_c^2 (y - l_B^2 k_x)^2 .$$

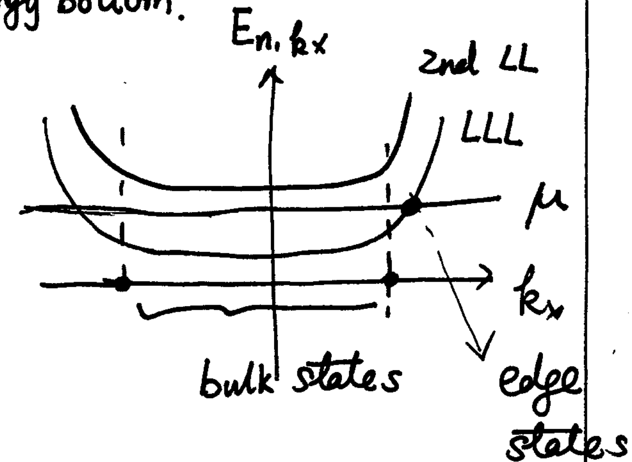
If we impose a boundry along the $y = \frac{L_y}{2}$, thus V_{k_x} is truncated to $+\infty$ at $y = \frac{L_y}{2}$.



For states $k_x < \frac{L_y}{2 l_B^2}$, its center of $V_{k_x}(y)$ is away from the boundary, thus it's not affected but the boundary. But as $k_x > \frac{L_y}{2 l_B^2}$, its bottom is cut by the boundary, and thus its energy is pushed up. Even at the classic level, we have

$$E_n(k_x) = \underbrace{\frac{1}{2} m \omega_c^2 (l_B^2)^2 \left(k_x - \frac{L_y}{2 l_B^2}\right)^2}_{\text{energy bottom.}} + (n + \tfrac{1}{2}) \hbar \omega$$

Thus we have the LL spectra with imposing boundaries at $-\frac{L_y}{2}$, and $\frac{L_y}{2}$.



☆ Chirality of the edge modes

The bulk states actually do not carry current. This is also in consistent with the classical picture — Electrons do cyclotron motion, and thus no charge transport. Now we explicity verify it.

$$j_x = \frac{1}{2m}\left[\psi^*\left(P_x - \frac{e}{c}A_x\right)\psi - \psi\left(P_x + \frac{e}{c}A_x\right)\psi^*\right]$$

For bulk state $\psi_{n,k_x}(x,y) = \phi_n(y - y_0(k))\,\dfrac{e^{ik_x x}}{\sqrt{L_x}}$

$$\left(P_x - \frac{e}{c}A_x\right)\psi_{n,k_x} = \left[\hbar k_x - \frac{eB}{c}y\right]\psi_{n,k_x}$$

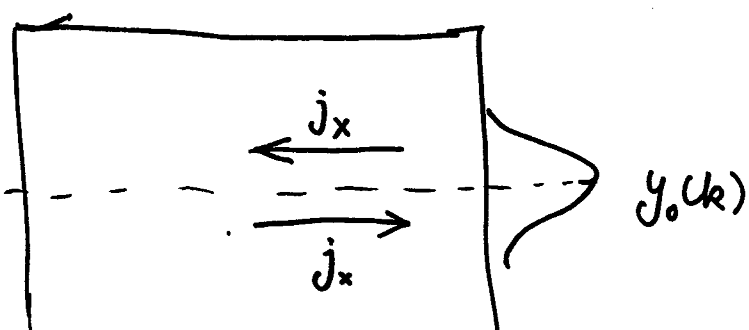
$$\left(P_x + \frac{e}{c}A_x\right)\psi^*_{n,k_x} = \left[-\hbar k_x + \frac{eB}{c}y\right]\psi^*_{n,k_x}$$

$$\Rightarrow \underset{(n,k_x)}{j_x} = \frac{1}{m}|\psi_{n,k_x}|^2\left[\hbar k_x - \frac{eB}{c}y\right] = \frac{\hbar}{m}|\psi_{n,k_x}|^2\left[y_0(k) - y\right]$$

$$\Rightarrow \underset{(n,k_x)}{I_x} = \int dy \cdot j_{x(n,k_x)} = \frac{\hbar}{m}\int dy\, |\psi_{n,k_x}|^2\left[y_0(k) - y\right]$$

$$= \frac{\hbar}{m}\int dy\, |\phi_n(y)|^2 (-)\,y = 0$$

even function odd



The ~~total~~ current carried by each state ψ_{n,k_x} is zero.

~~If imposed with a boundary, how about edge states?~~

let's consider the upper edge at $y = L_y/2$, in the limit $k_x \to +\infty$

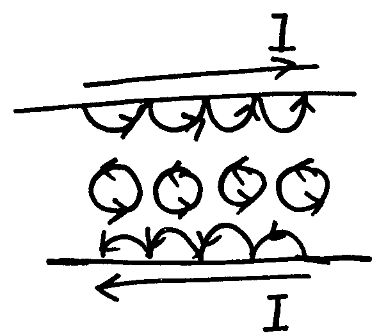
$$\psi_{n,k_x} = f_n(y)\,\frac{1}{\sqrt{L_x}}\,e^{ik_x x}, \quad \text{where } f_n(y) \to \delta(y - L_y/2)$$

$$\Rightarrow \left(P_x - \frac{e}{c}A_x\right)\psi_{n,k_x} \simeq \left(\hbar k_x - \frac{eB}{c}\frac{L_y}{2}\right)\psi_{n,k_x} \longrightarrow \hbar k_x \psi_{n,k_x}$$

$$j_{x,nk_x} \simeq \delta(y - L_y/2)\,\frac{\hbar k_x}{m}\,\frac{1}{\sqrt{L_x}} :$$

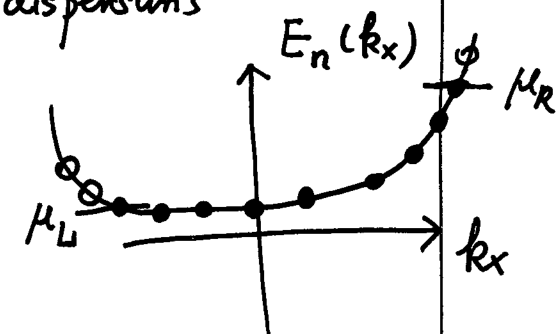
Edge does carry current!

classical picture



★ Now we calculate the Hall conductance

Now considering real sample, such that due to the effect of boundary and impurity, we cannot assume the Landau level energy is exactly flat, but with small dispersions



the group velocity

$$v_{k_x} = \frac{1}{\hbar}\frac{\partial E_n(k_x)}{\partial k_x}$$

$$j_{x,nk_x} = e|\psi_{n,k_x}|^2 v_x \Rightarrow I_{x,nk_x} = \int dy\, e|\psi_{n,k_x}|^2 v_x$$

$$= \frac{e}{L_x}\frac{1}{\hbar}\frac{\partial E_n(k_x)}{\partial k_x}$$

(assume $\psi_{n,k_x} = f_n(y)\dfrac{e^{ik_x x}}{\sqrt{L_x}}$ and $\int dy\,|f_n(y)|^2 = 1$)

$$\Rightarrow I_{x,n} = \sum_{\substack{k_x \\ \text{occupied}}} I_{x,n,k_x} = \frac{e}{L_x \hbar} L_x \int_{\text{occupied}} \frac{dk_x}{2\pi} \frac{\partial E(k_x)}{\partial k_x}$$

$$= \frac{e}{2\pi\hbar} \left[E(\text{right occupied}) - E(\text{left, occupied}) \right] = \frac{e}{h} \cdot (\mu_R - \mu_L)$$

$$= \frac{e^2}{h} \Delta V_y$$

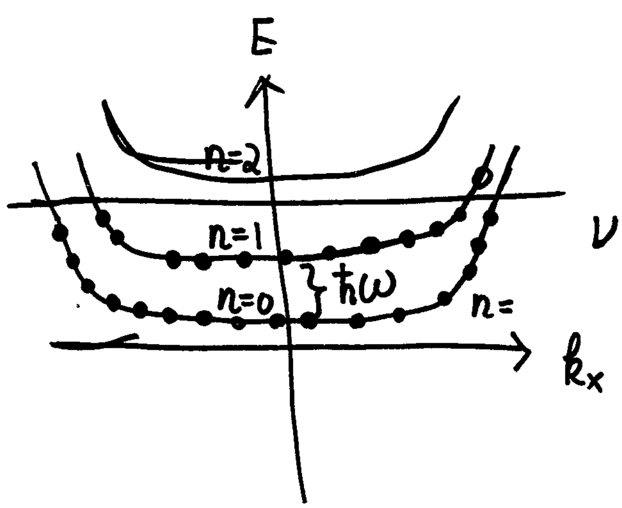
$\Rightarrow$ each Landau level contribute $\sigma_{xy,n} = \dfrac{e^2}{h}$

$\Rightarrow$ total Hall conductance

$$\boxed{\sigma_{xy} = \sum_{n,\text{occupied}} \sigma_{xy,n} = \frac{\nu e^2}{h}}$$

ν: filling number

Quantum Hall effect

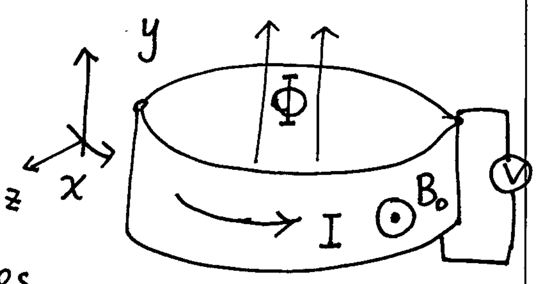


$\nu = 2$

04/04/2012

Laughlin argument for integer Qantum Hall

Goal: relate the current I carried around the loop (cylindrical geometry) to the voletage drop accross the two edges in the y-direction.



$$\vec{A} = -B_0 y \, \hat{e}_x$$

$$\Rightarrow \vec{B} = \partial_x A_y - \partial_y A_x = B_0 \, \hat{e}_z$$

Now let us thread a new flux Φ in the center of the cylinder.

The corresponding $\vec{A}' = \frac{\Phi}{L_x} \hat{e}_x$, where L_x is the circumstance arroung $\hat{e}_x$-direction.

The current I along the x-direction

$$I = c \frac{\partial E}{\partial \Phi} = \frac{c}{L_x} \frac{\partial E}{\partial A}$$

c.f. $\quad I = j_x \cdot L_y \quad \Rightarrow \frac{1}{c} j_x \cdot A \, (L_x L_y) = \Delta E$

For non-interaction electrons

$$H = \frac{1}{2m} \left(P_x - \frac{e}{c} A_x\right)^2 + \frac{1}{2m} P_y^2 + e E_0 y \quad \text{where } E_0 = \frac{\Delta V}{L_y}$$

$$\psi_{n,k_x}(\vec{r}) = e^{ikx} \phi_n(y - y_0(k))$$

$$\left[\frac{1}{2m} \left(\hbar k_x - \frac{eB}{c} y\right)^2 + \frac{P_y^2}{2m} + eE_0 y \right] \phi_n(y - y_0(k)) = E_n \phi_n(y - y_0(k))$$

$$\frac{m}{2}\omega^2\left[y-\frac{\hbar c}{eB}k_x\right]^2+eE_0 y=\frac{m}{2}\omega^2\left[y-\frac{\hbar c}{eB}k_x+\frac{eE_0}{m\omega^2}\right]^2$$

$$\boxed{\omega=\frac{eB}{mc}} \qquad +\frac{\hbar c}{eB}k_x eE_0-\frac{1}{2}\frac{e^2E_0^2}{m\omega^2}$$

$$=\frac{m}{2}\omega^2\left[y-y_0(k_x)\right]^2+eE_0\underbrace{\left[\frac{\hbar c}{eB}k_x-\frac{eE_0}{m\omega^2}\right]}_{y_0(k_x)}+\underbrace{\frac{1}{2}\frac{eE_0}{m\omega^2}}$$

$$\frac{m}{2}\left[\frac{eE_0}{m\omega}\right]^2=\frac{m}{2}\left[\frac{cE_0}{B}\right]^2$$

$$=\frac{m}{2}\omega^2\left[y-y_0(k_x)\right]^2+eE_0\,y_0(k_x)+\frac{mc^2}{2}\left[\frac{E_0}{B}\right]^2$$

$$y_0(k_x)=\frac{\hbar c}{eB}k_x-\frac{eE_0}{m\omega^2}=\frac{1}{eB/mc}\left[\frac{\hbar k_x}{m}-\frac{E_0}{B_0}c\right]=\frac{1}{\omega}\left[\frac{\hbar k_x}{m}-c\frac{E_0}{B_0}\right]$$

$$\Rightarrow\quad E_n=(n+1/2)\hbar\omega+eE_0\,y_0(k_x)+\frac{1}{2}mc^2\left(\frac{E_0}{B_0}\right)^2$$

Let us thread the extra flux $\Delta A\hat{e}_x=\dfrac{\Phi}{L_x}\hat{e}_x \Rightarrow A=B_0\left(y+\dfrac{\Delta A}{B_0}\right)$

$$\Rightarrow\quad \frac{m}{2}\omega^2\left[y-\frac{\hbar c}{eB}k_x+\frac{\Delta A}{B_0}\right]^2+eE_0 y$$

$$=\frac{m}{2}\omega^2\left(y-\frac{\hbar c}{eB}k_x+\frac{eE_0}{m\omega^2}+\frac{\Delta A}{B_0}\right]^2+eE_0\left[\frac{\hbar c}{eB_0}k_x-\frac{eE_0}{m\omega^2}-\frac{\Delta A}{B_0}\right]$$

$$=\frac{m}{2}\omega^2\left[y-y_0'(k_x)\right]^2+eE_0\,y_0'(k_x)+\frac{m\omega^2}{2}\left[\frac{eE_0}{m\omega^2}\right]^2$$

$$+\frac{m}{2}c^2\left[\frac{E_0}{B_0}\right]^2$$

with $y_0'(k_x)=y_0(k_x)-\dfrac{\Delta A}{B_0}$

~~Let chang $\Delta A \cdot L_x = \Delta\phi = \frac{hc}{e}$, the system comes back to~~ itself. The energy change is due to pump a particle between two edges

$$\Delta E = eV \Rightarrow I = c\,\frac{eV}{hc/e} = \frac{e^2}{h}V \Rightarrow \sigma_H = \frac{e^2}{h}.$$

If there are several sets of Landau level $\Rightarrow$ $\sigma_H = \frac{ne^2}{h}$.

✡ For dirty and interacting systems

As long as, the Fermi surface lies inside the gap, thread a $\Delta\phi = \frac{hc}{e}$ adiabatically cannot generate a particle-hole excitation across the gap, thus the bulk should remain the same as ~~to~~ before flux threading. The only possible place to change is the edge, which is gapless. It removes a particle from one edge, moving it into the bulk, and extra another one from the bulk to another edge. $\longrightarrow$ as a pump.