§2. Primary fields & correlation functions

under transformation $z \to f(z)$ / $\bar{z} \to \bar{f}(\bar{z})$, the measure changes as

$$ds^2 = dz d\bar{z} \longrightarrow ds'^2 = df d\bar{f} = \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right) dz d\bar{z}$$

for field $\phi(z, \bar{z})$, we define it transforms as $\to \phi'(f, \bar{f})$

$$\phi(z\bar{z}) (dz)^h (d\bar{z})^{\bar{h}} = \phi'(f, \bar{f}) (df)^h (d\bar{f})^{\bar{h}}$$

$$\Rightarrow \boxed{\phi'(f(z), \bar{f}(\bar{z})) = \left(\frac{\partial f}{\partial z}\right)^{-h} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z\bar{z})}$$

primary field if for all the local conformal transf

quasi-primary field for the global conf ~~trans~~ transf /

for infinitismall transformation, $z \to z + \varepsilon(z)$
$\bar{z} \to \bar{z} + \bar{\varepsilon}(\bar{z})$

we have $\left(\frac{\partial f}{\partial z}\right)^{-h} \simeq (1 - h \partial_z \varepsilon)$

$$\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{-\bar{h}} \simeq (1 - \bar{h} \partial_{\bar{z}} \bar{\varepsilon})$$

$\Rightarrow$ the variation of the quasi-primary field

$$\delta_{\varepsilon\bar{\varepsilon}} \phi = \phi'(z, \bar{z}) - \phi(z, \bar{z}) = (1 - h\partial\varepsilon)(1 - \bar{h}\bar{\partial}\bar{\varepsilon})\, \phi(z - \varepsilon, \bar{z} - \bar{\epsilon}) - \phi(z, \bar{z})$$

$$\simeq (1 - h\partial_z\varepsilon - \bar{h}\partial_{\bar{z}}\bar{\varepsilon})(1 - \epsilon\partial_z - \bar{\epsilon}\partial_{\bar{z}})\, \phi(z, \bar{z}) - \phi(z, \bar{z})$$

$$\boxed{\delta_{\varepsilon\bar{\varepsilon}} \phi = -(h\partial_z\varepsilon + \varepsilon\partial_z + \bar{h}\partial_{\bar{z}}\bar{\varepsilon} + \bar{\varepsilon}\partial_{\bar{z}})\, \phi(z, \bar{z})}$$

please note the location inside ϕ' is $(z, \bar{z})$, thus when we use the quasi-primary field condition, we need to shift the $z \to z - \varepsilon$.

Symmetry transformation and correlation functions.

① consider under the transformation $x \to x'$, the field $\phi(x)$ ~~transforms~~ as $\phi(x) \to \phi'(x') = F[\phi(x)]$, where $F[\cdot]$ is a functional. For example, for the scale transformation $x' = \lambda x$, we define $\phi'(x') = \phi'(\lambda x) = \lambda^{-\Delta} \phi(x)$.

Let us consider the free field action $S[\phi] = \int d^d x \, \partial_\mu \phi \, \partial^\mu \phi$.

Check: field $\phi(x) \xrightarrow[\text{scale transform}]{} \underset{\uparrow \text{ new field}}{\phi'(x)} = \phi'(\lambda \lambda^{-1} x) = \lambda^{-\Delta} \phi(\lambda^{-1} x)$

$$S[\phi'] = \int d^d x \, \partial_\mu \phi'(x) \, \partial^\mu \phi'(x) = \lambda^{-2\Delta} \int d^d x \, \partial_\mu \phi(\lambda^{-1} x) \, \partial^\mu \phi(\lambda^{-1} x)$$

$$= \lambda^{-2\Delta} \lambda^{d-2} \int d^d x \, \partial_\mu \phi(x) \, \partial^\mu \phi(x) = \lambda^{2[\frac{d}{2} - 1 - \Delta]} \, S[\phi]$$

we set $\Delta = d/2 - 1$ to maintain the scale invariance.

② Correlation function

$$\langle \phi(x_1') \phi(x_2') \rangle = \frac{1}{Z} \int [d\phi] \, \phi(x_1') \, \phi(x_2') \, e^{-S[\phi]} \quad \text{(by definition)}$$

$$= \frac{1}{Z} \int [d\phi'] \, \phi'(x_1') \, \phi'(x_2') \, e^{-S[\phi']} \quad \text{(change dumy variable)}$$

symmetry assumed

$$= \frac{1}{Z} \int [d\phi] \, F[\phi(x_1)] \, F[\phi(x_2)] \, e^{-S[\phi]} \quad \text{(functional variable transf)}$$

(assuming Jocabian is trivial, i.e. doesn't depend on ϕ. This may be violet and leads to anomaly)!

$$\Rightarrow \boxed{\langle \phi_1(x_1') \cdots \phi_n(x_n') \rangle = \langle F[\phi_1(x_1)] \cdots F[\phi_n(x_n)] \rangle}$$

Similarly

$$\langle \phi_1(x_1)\, \phi_2(x_2) \rangle = \frac{1}{Z} \int D[\phi]\; \phi_1(x_1)\, \phi_2(x_2)\; e^{-S[\phi]}$$ change dummy

Jacobian trivial

$$= \frac{1}{Z} \int D[\phi']\; \phi_1'(x_1)\, \phi_2'(x_2)\; e^{-S[\phi']}$$ symmetry

$$= \frac{1}{Z} \int D[\phi]\; \phi_1'(x_1)\, \phi_2'(x_2)\; e^{-S[\phi]}$$

i.e. $$\boxed{\langle \phi_1(x_1)\, \phi_2(x_2) \cdots \rangle = \langle \phi_1'(x_1) \cdots \phi_n'(x_n) \rangle}$$

As long as fix the location $x_1, x_2 \cdots x_n$, the correlation functions are the same no matter using old or new variables. (

* Constraint on two-point correlation functions

$$G^{(2)}(z_1\bar{z}_1; z_2\bar{z}_2) = \langle \phi_1(z_1\bar{z}_1)\, \phi_2(z_2\bar{z}_2) \rangle$$

$$\delta G^{(2)}(z_1\bar{z}_1; z_2\bar{z}_2) = \langle \phi_1'(z_1\bar{z}_1)\, \phi_2'(z_2\bar{z}_2) \rangle - \langle \phi_1(z_1\bar{z}_1)\, \phi_2(z_2\bar{z}_2) \rangle = 0$$

$$\phi'(z\bar{z}) = \phi(z\bar{z}) + \delta_{\epsilon\bar{\epsilon}}\phi \quad \Rightarrow$$

$$\langle \delta_{\epsilon\bar{\epsilon}}\phi_1,\ \phi_2 \rangle + \langle \phi_1,\ \delta_{\epsilon\bar{\epsilon}}\phi_2 \rangle = 0$$

$$\left[h_1 \partial_{z_1}\epsilon(z_1) + \epsilon(z_1)\partial_{z_1} + h_2 \partial_{z_2}\epsilon(z_2) + \epsilon(z_2)\partial_{z_2} \right] + \begin{pmatrix} z_{1,2} \to \bar{z}_{1,2} \\ h_{1,2} \to \bar{h}_{1,2} \end{pmatrix}$$

$$\cdot\, G^{(2)}(z_1\bar{z}_1, z_2\bar{z}_2) = 0$$

• translation $\epsilon(z) = \epsilon \Rightarrow$

$$(\partial_{z_1} + \partial_{z_2} + \partial_{\bar{z}_1} + \partial_{\bar{z}_2})\, G^{(2)}(z_1\bar{z}_1, z_2\bar{z}_2) = 0 \Rightarrow$$

$$G^{(2)}(z_1\bar{z}_1\,;\,z_2\bar{z}_2) = G^{(2)}(z_{12}\,;\,\bar{z}_{12})$$

B: rotation & dilation $\quad \varepsilon(z) = \varepsilon z \Rightarrow$

$$\left\{ z_1\partial_1 + h_1 + z_2\partial_2 + h_2 + (z_{12}\to\bar{z}_{12},\ h_{1,2}\to\bar{h}_{1,2}) \right\} G^{(2)} = 0$$

$$\left[z_{12}\partial_{12} + h_1 + h_2 + h.c. \right] G^{(2)}(z_{12},\bar{z}_{12}) = 0$$

$$G^{(2)}(z_{12},\bar{z}_{12}) = \frac{C_{12}}{z_{12}^{\,h_1+h_2}\ \bar{z}_{12}^{\,\bar{h}_1+\bar{h}_2}}$$

C: special conf $\quad \varepsilon(z) = \varepsilon z^2$

$$\Rightarrow \left\{ \left[z_1^2\partial_1 + 2h_1z_1 + z_2^2\partial_2 + z_2^2\partial_2 + 2h_2z_2 \right] + c.c. \right\} G^{(2)} = 0$$

$$z_1^2\partial_1 + z_2^2\partial_2 = (z_1^2+z_2^2)\frac{\partial_{z_1+z_2}}{2} + (z_1^2-z_2^2)\,\partial_{12}$$

$$\Rightarrow \left\{ (z_1^2-z_2^2)\,\partial_{12} + 2(h_1z_1+h_2z_2) + c.c. \right\} G^{(2)} = 0$$

plug in $\quad z_{12}\,\partial_{12}\,G^{(2)} = -(h_1+h_2)\,G^{(2)}$

$$\Rightarrow \left[-(z_1+z_2)(h_1+h_2) + 2(h_1z_1+h_2z_2) + c.c. \right] G^{(2)} = 0$$

$$\left\{ (h_1-h_2)(z_1-z_2) + h.c. \right\} G^{(2)} = 0 \Rightarrow h_1 = h_2,\quad \bar{h}_1 = \bar{h}_2$$

$$\boxed{G^{(2)}(z_{12},\bar{z}_{12}) = \frac{C_{12}}{z_{12}^{2h}\ z_{12}^{2\bar{h}}}}$$

C_{12} can be absorbed as normalizat factor.

* scaling dimension $\quad \Delta = h + \bar{h}$, under dilation $\quad z \to \lambda z$, $\ \bar{z} \to \lambda \bar{z}$

$$G^{(2)} \to \lambda^{-(h+\bar{h})\times 2}\ G^{(2)}$$

* Rotation $z \to e^{i\theta} z$, $\bar{z} \to e^{-i\theta}\bar{z}$

$$G^{(2)} \to e^{-i\theta(h-\bar{h})\times 2}\, G^{(2)}$$ ← conformal spin

3-point correlation function

translation invariant

$$G^{(3)}(z_1 z_2 z_3 ; z_1' z_2' z_3') = \langle \phi(z_1\bar{z}_1)\phi(z_2\bar{z}_2)\phi(z_3\bar{z}_3)\rangle = \sum_{abc} \frac{C_{123}^{abc}\, C_{123}^{\bar{a}\bar{b}\bar{c}}}{(z_{12}^{a} z_{23}^{b} z_{31}^{c})\cdot(\bar{z}_{12}^{\bar{a}} \bar{z}_{23}^{\bar{b}} \bar{z}_{31}^{\bar{c}})}$$

- check rotation and dilation

$$\left[z_1\partial_1 + z_2\partial_2 + z_3\partial_3 + h_1 + h_2 + h_3 + c.c.\right] G^{(3)} = 0$$

for functions as $\frac{1}{z_{12}^{a} z_{23}^{b} z_{31}^{c}}$, the effect $z_1\partial_1 + z_2\partial_2 + z_3\partial_3$

$$\frac{-a z_1}{z_{12}^{a+1} z_{23}^{b} z_{31}^{c}} + \frac{c z_1}{z_{12}^{a} z_{23}^{b} z_{31}^{c+1}} + \frac{a z_2}{z_{12}^{a+1} z_{23}^{b} z_{31}^{c}} + \frac{-b z_2}{z_{12}^{a} z_{23}^{b+1} z_{31}^{c}}$$

$$+ \frac{b z_3}{z_{12}^{a} z_{23}^{b+1} z_{31}^{c}} + -\frac{c z_3}{z_{12}^{a} z_{23}^{b} z_{31}^{c+1}} = \left[-a-b-c\right]\frac{1}{z_{12}^{a} z_{23}^{b} z_{31}^{c}}$$

$$\Rightarrow \left[-(a+b+c) + (h_1+h_2+h_3)\right] = 0$$

similarly $\bar{a}+\bar{b}+\bar{c} = \bar{h}_1 + \bar{h}_2 + \bar{h}_3$

- Check special conformal transformation

$$\left[z_1^2\partial_1 + 2h_1 z_1 + z_2^2\partial_2 + 2h_2 z_2 + z_3^2\partial_3 + 2h_3 z_3 + c.c.\right] G^{(3)} = 0$$

again when applying to the function such as $\frac{1}{z_{12}^{a} z_{23}^{b} z_{31}^{c}}$

$$z_1^2\partial_1 + z_2^2\partial_2 + z_3^2\partial_3 = \frac{-a z_1^2}{z_{12}^{a+1} z_{23}^{b} z_{31}^{c}} + \frac{c z_1^2}{z_{12}^{a} z_{23}^{b} z_{31}^{c+1}} + \frac{a z_2^2}{z_{12}^{a+1} z_{23}^{b} z_{31}^{c}} + \frac{-b z_2^2}{z_{12}^{a} z_{13}^{b+1} z_{31}^{c}}$$

$$+\frac{b z_3^2}{z_{12}^a z_{23}^{b+1} z_{31}^c} + \frac{-c z_3^2}{z_{12}^a z_{23}^b z_{31}^{c+1}} = \left[(z_1+z_2)(-a) + (z_2+z_3)(-b) + (z_3+z_1)(-c)\right] \cdot \frac{1}{z_{12}^a z_{23}^b z_{31}^c}$$

$$\Rightarrow \left\{ (-a)(z_1+z_2) + (-b)(z_2+z_3) + (-c)(z_3+z_1) + 2h_1 z_1 + 2h_2 z_2 + 2h_3 z_3 \right\} G^3 = 0$$

$$\Rightarrow \begin{cases} -a-c+2h_1=0 \\ -b-a+2h_2=0 \\ -c-b+2h_3=0 \end{cases} \Rightarrow \begin{aligned} a &= h_1+h_2-h_3 \\ b &= h_2+h_3-h_1 \\ c &= h_3+h_1-h_2 \end{aligned}$$

$$G^{(3)}(z_1 z_2 z_3 ; \bar z_1 \bar z_2 \bar z_3) = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{31}^{h_3+h_1-h_2}} \otimes \frac{\bar C_{123}}{\bar z_{12}^{\bar h_1+\bar h_2-\bar h_3} \bar z_{23}^{\cdots} \bar z_{31}^{\cdots}}$$

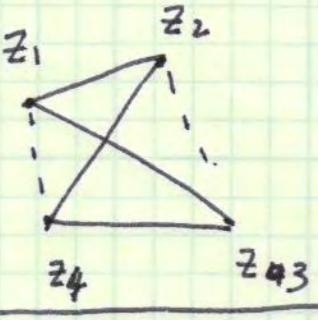
* The reason why $G^{(3)}$ can be constrained to such a form is that for any set of (z_1, z_2, z_3), we can define a conformal transf

$$z \to \frac{z-z_3}{-z+z_1} \, \frac{z_1-z_2}{z_2-z_3}$$

such that $(z_1 z_2 z_3) \to (\infty, 1, 0)$, $\Rightarrow$ no other degrees of freedom

Thus the 3-points correlation function can be constrained by a parameter.

* Four point function



$z_1\ z_2\ z_3\ z_4 \longrightarrow (0, 1, \infty, x)$

following $z \to \dfrac{z - z_3}{-z + z_1} \cdot \dfrac{z_1 - z_2}{z_2 - z_3}$

with $x = \dfrac{z_{12} z_{34}}{z_{13} z_{24}}$: cross ratio ;

$$\boxed{x = \frac{\left(\frac{1}{z_1} - \frac{1}{z_2}\right)\left(\frac{1}{z_3} - \frac{1}{z_4}\right)}{\left(\frac{1}{z_1} - \frac{1}{z_3}\right)\left(\frac{1}{z_2} - \frac{1}{z_4}\right)} \text{ is invariant under special conformal transf.}}$$

$$y = \frac{z_{12} z_{34}}{z_{14} z_{23}} = \frac{x}{1-x} \quad \Leftarrow \quad \text{check } \frac{1}{x} - \frac{1}{y} = 1$$

we can try $\langle \phi_1(z_1 \bar{z}_1)\, \phi_2(z_2 \bar{z}_2)\, \phi_3(z_3 \bar{z}_3)\, \phi_4(z_4 \bar{z}_4) \rangle = f(x, \bar{x}) (z_{12}^{-a} z_{13}^{-b} z_{14}^{-c} z_{23}^{-d} z_{24}^{-e} z_{34}^{-f}$

$\otimes (\cdots\cdots)$

check rotation

$$\left(\sum_i z_i \partial_i + h_i\right) G^4 = 0$$

$$\left(\sum_i z_i \partial_i\right) \left[z_{12}^{-a} z_{13}^{-b} z_{14}^{-c} z_{23}^{-d} z_{24}^{-e} z_{34}^{-f}\right] =$$ 依次作用于每项 z_{ij}

$$= (-a-b-c-d-e-f)\left[z_{12}^{-a} \cdots \right] \Rightarrow$$

$$a+b+c+d+e+f = h_1 + h_2 + h_3 + h_4$$

check conformal $\quad (z_i^2 \partial_i + 2h_i z_i)\, G^4 = 0$

$$\left(\sum_i z_i^2 \partial_i\right)\left[z_{12}^{-a} z_{13}^{-b} z_{14}^{-c} z_{23}^{-d} z_{24}^{-e} z_{34}^{-f}\right] = -(z_1^2 - z_2^2)\, a\, z_{12}^{-a+1} z_{13}^{-b} z_{14}^{-c} z_{23}^{-d} z_{24}^{-e} z_{34}^{-f}$$

$+ \cdots\cdots$

$+ \cdots$

$$= \Big[(z_1+z_2)(-a) + (z_1+z_3)(-b) + (z_1+z_4)(-c)$$
$$(z_2+z_3)(-d) + (z_2+z_4)(-e) + (z_3+z_4)(-f)\Big]\, G^4$$

Combine with $+\ 2h_i z_i \quad G^4 \quad = 0$

$$\Rightarrow \left.\begin{aligned} a+b+c &= 2h_1 \\ a+d+e &= 2h_2 \\ b+d+f &= 2h_3 \\ c+e+f &= 2h_4 \end{aligned}\right\}$$

define $h = h_1 + h_2 + h_3 + h_4$

$$\Rightarrow a+b+c+d+e+f = h$$

$\Rightarrow$ there are six-varibles, which is not sufficient to detemine all the variables. but only 4 equatiuns

The results are
$$a = h_1+h_2-h/3 \quad c = h_1+h_4-h/3 \quad e = h_2+h_4-h/3$$
$$b = h_1+h_3-h/3 \quad d = h_2+h_3-h/3 \quad f = h_3+h_4-h/3$$

$\rightarrow$ prove we should $\underbrace{b+e = c+d = a+f}$ (?)

$$\boxed{\langle \phi_1(z_1\bar{z}_1)\, \phi_2(z_2\bar{z}_2)\, \phi_3(z_3\bar{z}_3)\, \phi_4(z_4\bar{z}_4) \rangle = f(x,\bar{x}) \prod_{i>j} z_{ij}^{-(h_i+h_j-h/3)} \prod_{i>j} \bar{z}_{ij}^{-(\bar{h}_i+\bar{h}_j-\bar{h}/3)}}$$

(*) Comment: the four constraint's cannot fully solve $a, b, \cdots f$. We just chouse the solutions as above. The other degree of freedom is absorbed in $f(x,\bar{x})$.

For example: $x = \dfrac{z_{12} z_{34}}{z_{13} z_{24}}$. Multifying x will change $\begin{cases} b \to b+1, & a \to a-1 \\ e \to e+1, & f \to f-1 \end{cases}$

$y = \dfrac{z_{12} z_{34}}{z_{14} z_{23}} = \dfrac{x}{1-x}$, Multifying y change $\begin{cases} c \to c+1, & a \to a-1 \\ d \to d+1, & f \to f-1. \end{cases}$

Thus the above solution is generic!

* how to understand the constraint from CF invariance to correlation function? (direct check!)

$$\langle \phi_1(z_1'\bar{z}_1')\,\phi_2(z_2'\bar{z}_2')\rangle = \left(\frac{\partial z_1'}{\partial z_1}\right)^{-h_1}\left(\frac{\partial z_2'}{\partial z_2}\right)^{-h_2}\left(\frac{\partial \bar{z}_1'}{\partial \bar{z}_1}\right)^{-\bar{h}_1}\left(\frac{\partial \bar{z}_2'}{\partial \bar{z}_2}\right)^{-h_2}\langle \phi_1(z_1\bar{z}_1)\cdot\phi_2(z_2\bar{z}_2)\rangle$$

① set $z_1' = \lambda z_1$, $z_2' = \lambda z_2 \Rightarrow$

$$\lambda^{-(h_1+h_2)}\,\lambda^{-(\bar{h}_1+\bar{h}_2)}$$

$$\langle \phi_1(\lambda z_1\,\overline{\lambda z_1}),\,\phi_2(\lambda z_2\,\overline{\lambda z_2})\rangle = \quad \cdot\,\langle \phi_1(z_1\bar{z}_1)\,\phi_2(z_2\bar{z}_2)\rangle$$

↖ rotation & dilation

② set $z_1' = \dfrac{z}{cz+1}$, or $\dfrac{1}{z'} = \dfrac{1}{z} + c$

special conf

$$\frac{\partial z'}{\partial z} = \frac{z'^2\,\partial(1/z')}{z^2\,\partial(1/z)} = \frac{z'^2}{z^2} \leftarrow$$

$$\Rightarrow \langle \phi_1(z_1'\bar{z}_1')\,\phi_2(z_2'\bar{z}_2')\rangle = \left(\frac{z_1'}{z_1}\right)^{-2h_1}\left(\frac{z_2'}{z_2}\right)^{-2h_2}\left(\frac{\bar{z}_1'}{\bar{z}_1}\right)^{-2\bar{h}_1}\left(\frac{\bar{z}_2'}{\bar{z}_2}\right)^{-2\bar{h}_2}\langle \phi_1(z_1\bar{z}_1)\,\phi_2(z_2\bar{z}_2)\rangle$$

Apparently $\langle \phi_1(z_1\bar{z}_1)\,\phi_2(z_2\bar{z}_2)\rangle = \dfrac{C_{12}}{z_{12}^{2h}\,\bar{z}_{12}^{2\bar{h}}}$ satisfies ①

Now let us check ②

$$\langle \phi_1(z_1\bar{z}_1)\,\phi_2(z_2\bar{z}_2)\rangle = \frac{(z_1z_2)^{-2h}(\bar{z}_1\bar{z}_2)^{-2\bar{h}}\,C_{12}}{\left(\frac{1}{z_1}-\frac{1}{z_2}\right)^{2h}\left(\frac{1}{\bar{z}_1}-\frac{1}{\bar{z}_2}\right)^{2\bar{h}}}$$

(set $h_1 = h_2 = h$, $\bar{h}_1 = \bar{h}_2 = \bar{h}$)

$$\Rightarrow \langle \phi_1(z_1'\bar{z}_1')\,\phi_2(z_2'\bar{z}_2')\rangle = \frac{(z_1'z_2')^{-2h}(\bar{z}_1'\bar{z}_2')^{-2\bar{h}}\,C_{12}}{\left(\frac{1}{z_1'}-\frac{1}{z_2'}\right)^{2h}\left(\frac{1}{\bar{z}_1}-\frac{1}{\bar{z}_2}\right)^{2\bar{h}}}$$

(✓)

2 Now we check 3-point correlation func?

$$\langle \phi_1(z_1'\bar{z}_1')\,\phi_2(z_2\bar{z}_2')\,\phi_3(z_3\bar{z}_3')\rangle = \prod_i \left(\frac{\partial z_i'}{\partial z_i}\right)^{-h_i}\left(\frac{\partial \bar{z}_i'}{\partial \bar{z}_i}\right)^{-\bar{h}_i} \langle \phi(z_1\bar{z}_1)\,\phi(z_2\bar{z}_2)\,\phi(z_3\bar{z}_3)\rangle$$

① set $z_i' = \lambda z_i$, $\bar{z}_i' = \bar{\lambda} z_i'$

$$\boxed{\langle \phi_1(\lambda z_1\, \bar{\lambda}\bar{z}_1)\,\phi_2(\lambda z_2,\bar{\lambda}\bar{z}_2)\,\phi_3(\lambda z_3\,\bar{\lambda}\bar{z}_3)\rangle = \lambda^{-(h_1+h_2+h_3)}\,\bar{\lambda}^{-(\bar{h}_1+\bar{h}_2+\bar{h}_3)} \langle \phi(z_1\bar{z}_1)\cdots\phi(z_3\bar{z}_3)\rangle}$$

apparently $\quad G^{(3)} \propto \dfrac{1}{z_{12}^{h_1+h_2-h_3}\, z_{23}^{h_2+h_3-h_1}\, z_{31}^{h_3+h_1-h_2}} \otimes \dfrac{1}{\cdots\cdots}$

satisfies this constraint. Morever, conformal is a stronger constraint

② $\dfrac{1}{z'} = \dfrac{1}{z} + C$

$$\boxed{\langle \phi_1(z_1'\bar{z}_1')\cdots\phi_3(z_3\bar{z}_3')\rangle = \prod_i \left(\frac{z_i'}{z_i}\right)^{-2h_i}\left(\frac{\bar{z}_i'}{\bar{z}_i}\right)^{-2h_i} \langle \phi_1(z_1\bar{z}_1)\cdots\phi_3(z_3\bar{z}_3)\rangle}$$

$$G^{(3)} \propto \frac{(z_1z_2)^{-(h_1+h_2-h_3)}\,(z_2z_3)^{-(h_2+h_3-h_1)}\,(z_3z_1)^{-(h_3+h_1-h_2)}}{\left(\frac{1}{z_1}-\frac{1}{z_2}\right)^{h_1+h_2-h_3}\left(\frac{1}{z_2}-\frac{1}{z_3}\right)^{h_2+h_3-h_1}\left(\frac{1}{z_3}-\frac{1}{z_1}\right)^{h_3+h_1-h_2}} \otimes \frac{\cdots}{\cdots}$$

$$\propto \frac{z_1^{-2h_1}\, z_2^{-2h_2}\, z_3^{-2h_3}}{\Pi\left(\frac{1}{z_1}-\frac{1}{z_2}\right)^{h_1+h_2-h_3}\left(\frac{1}{z_2}-\frac{1}{z_3}\right)^{h_2+h_3-h_1}\left(\frac{1}{z_3}-\frac{1}{z_1}\right)^{h_3+h_1-h_2}} \otimes \frac{\cdots}{\cdots}$$

It's easy to see the power in the numerator matches the requirement of conformal field theory

③ similarly for the special conformal transformation $\dfrac{1}{z'} = \dfrac{1}{z} + C$

$$\boxed{\langle \phi_1(z_1'\bar{z}_1')\cdots\phi_4(\bar{z}_4\bar{z}_4')\rangle = \prod_i \left(\frac{z_i'}{z_i}\right)^{-2h_i}\left(\frac{\bar{z}_i'}{z_i}\right)^{-2\bar{h}_i} \langle \phi_1(z_1\bar{z}_1)\cdots\phi_4(z_4 z_4')\rangle}$$

$$\cancel{G^{(4)} \propto f(x,\bar{x}) \prod_{i>j} \frac{(z_1z_2)^{-(h_1+h_2-h/3)}}{\left(\frac{1}{z_1}-\frac{1}{z_2}\right)^{\underbrace{h_1+h_2-h/3}_{a}}} \frac{(z_1z_3)^{-(h_1+h_2-h/3)}}{\left(\frac{1}{z_1}-\frac{1}{z_3}\right)^{\underbrace{h_1+h_2-h/3}_{b}}} \cdots}$$

It's easy to see that its consistent with the previos calculation.

$$\begin{cases} a+b+c = 2h_1 \\ a+d+e = 2h_2 \\ b+d+f = 2h_3 \\ c+e+f = 2h_4 \end{cases}$$

$$G^{(4)} \propto f(x,\bar{x}) \frac{(z_1 z_2)^{-a}}{\left(\frac{1}{z_1}-\frac{1}{z_2}\right)^a} \frac{(z_1 z_3)^{-b}}{\left(\frac{1}{z_1}-\frac{1}{z_3}\right)^b} \frac{(z_1 z_4)^{-c}}{\left(\frac{1}{z_1}-\frac{1}{z_4}\right)^c} \frac{(z_2 z_3)^{-d}}{\left(\frac{1}{z_2}-\frac{1}{z_3}\right)^d} \frac{(z_2 z_4)^{-e}}{\left(\frac{1}{z_2}-\frac{1}{z_4}\right)^e} \frac{(z_3 z_4)^{-f}}{\left(\frac{1}{z_3}-\frac{1}{z_4}\right)^f}$$