

# Bethe ansatz — spin chain

①

Wavefunction

$$\psi(x_1, \dots, x_M) = \sum_P A_P e^{i(k_{p_1} x_1 + k_{p_2} x_2 + \dots + k_{p_M} x_M)}$$

Consider 2 permutations connected by a nearest neighbour exchange

$$P: (k_{p_1} \dots k_{p_j} = k, k_{p_{j+1}} = k', \dots) \leftarrow A_P$$

$$P': (k_{p'_1} \dots k_{p'_j} = k', k_{p'_{j+1}} = k, \dots) \leftarrow A_{P'}$$

$$A_{P'}/A_P \equiv -e^{i\theta(k', k)} = -\frac{e^{i(k+k') - 2\Delta e^{ik'+1}}}{e^{i(k+k') - 2\Delta e^{ik+1}}}$$

The BA equations

$$e^{ik_j N} = (-1)^{M-1} e^{i \sum_{\ell=1}^M \theta(k_j, k_\ell)} \quad j=1, 2, \dots, M$$

and  $E = J \sum_{j=1}^M (\cos k_j - \Delta)$

For  $\Delta=1$ , parameterization

$$e^{ik_j} = \frac{\lambda_j + i/2}{\lambda_j - i/2} \Rightarrow \lambda_j = \frac{1}{2} \cot \frac{k_j}{2}$$

then  $e^{i\theta(k_j, k_\ell)} = -\frac{\lambda_j - \lambda_\ell + i}{\lambda_j - \lambda_\ell - i}$

§ Heisenberg limit ( $\Delta=1$ )

$$e^{ik_j N} = \prod_{\ell \neq j} \frac{\lambda_j - \lambda_\ell + i}{\lambda_j - \lambda_\ell - i} \quad \text{where } \lambda_j = \frac{1}{2} \cot \frac{k_j}{2}$$

define  $\phi_{j,\ell}$  as  $e^{i\phi_{j,\ell}} = \frac{\lambda_j - \lambda_\ell + i}{\lambda_j - \lambda_\ell - i} \Rightarrow \lambda_j - \lambda_\ell = \cot \frac{1}{2} \phi_{j,\ell} \quad (j \neq \ell)$

$$\Rightarrow N k_j = 2\pi m_j + \sum_{\ell \neq j} \phi_{j,\ell} \quad m_j \text{'s are integers mod } N$$

$$\left\{ \begin{aligned} 2 \cot \frac{1}{2} \phi_{j,\ell} &= \cot \frac{k_j}{2} - \cot \frac{k_\ell}{2} \\ &\in \{1, 3, \dots, N-1\}. \end{aligned} \right.$$

Then since  $\phi_{j,\ell} = -\phi_{\ell,j} \Rightarrow \sum_{j=1}^M k_j = \frac{2\pi}{N} \sum_{j=1}^M m_j \rightarrow \left(\frac{N}{2}\right)^2$

$$\sum_{j=1}^M k_j = \frac{2\pi}{N} \sum_{j=1}^M m_j \rightarrow \left(\frac{N}{2}\right)^2$$

$$\sum_{j=1}^M k_j = \frac{2\pi}{4} M = \pi N/2$$

\* Ground state pattern for  $\{m_j\} = \{1, 3, \dots, N-1\}$

there're no two  $m$ 's adjacent to each other, — no bound states.

The key is to solve the distribution function of  $k$ . For this

purpose, we define continuous variables  $x = \frac{2j-1}{N}$  and  $y = \frac{2\ell-1}{N}$ .

For  $m_j = 2j-1$ ,  $\rightarrow$  rescale to  $0 < x < 1$ . The Bethe Eq is reduced to

$$k(x) = 2\pi x + \frac{1}{2} \int_0^1 dy \phi(k(x), k(y)), \quad \text{where}$$

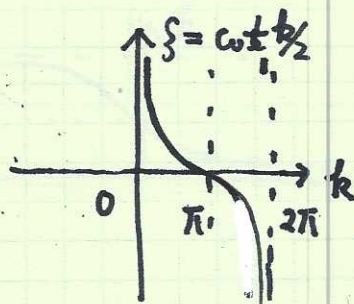
$$\cot \frac{\phi(k(x), k(y))}{2} = \frac{1}{2} \cot \frac{k(x)}{2} - \frac{1}{2} \cot \frac{k(y)}{2}$$

Then  $E = J \sum_{j=1}^{N/2} (\cos k_j - 1) = \frac{N}{2} J \int_0^1 dx (\cos k(x) - 1)$

$$\sum_j \rightarrow \int \frac{dx}{\frac{2}{N}} = \frac{N}{2} \int dx, \quad \sum_\ell \rightarrow \frac{N}{2} \int dy$$

Set  $\xi = \cot \frac{k(x)}{2}$

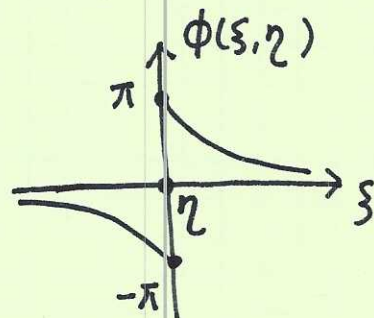
as  $k: 0 \rightarrow \pi \rightarrow 2\pi$   
 $\xi: +\infty \rightarrow 0 \rightarrow -\infty$



Then  $\cot \frac{1}{2} \phi(\xi, \eta) = \frac{1}{2} [\xi - \eta]$  with  $\begin{cases} \xi = \cot \frac{k(x)}{2} \\ \eta = \cot \frac{k(y)}{2} \end{cases}$

Let us fix  $\eta$ :

- $\xi \rightarrow \infty \iff \phi = 0$
- $\xi \rightarrow \eta + 0^+ \implies \phi = \pi$
- $\xi \rightarrow \eta + 0^- \implies \phi = -\pi$
- $\xi \rightarrow -\infty \implies \phi = 0$



$\Rightarrow k(x) = 2\pi x + \int_0^1 dy \cot^{-1} \frac{\xi(x) - \eta(y)}{2}$  ← with the convention that  $\cot^{-1} x$  change sign at  $x=0$

$\frac{dk(x)}{dx} = 2\pi + \int_0^1 dy \left[ \frac{-1}{1 + (\xi - \eta)^2/4} + 2\pi \delta(\xi - \eta) \right] \frac{1}{2} \frac{d\xi}{dx}$   
 due to the discontinuity

The delta-fun part  $\int_0^1 dy \delta(\xi - \eta) \frac{d\xi}{dx} = \int_0^1 dy \delta(\xi(x) - \eta(y)) \frac{d\xi}{dx}$

when  $\xi(x) = \eta(y)$  ← please notice that  $\xi$  and  $\eta$  are same function

$\Rightarrow x=y, \Rightarrow \delta(\xi(x) - \eta(y)) = \frac{\delta(x-y)}{|d\eta/dy|} = \frac{\delta(x-y)}{|d\xi/dx|}$

Since  $\xi$  is monotonically decreasing func,  $\Rightarrow d\xi/dx < 0$

$\Rightarrow \int_0^1 dy \delta(\xi(x) - \eta(y)) \frac{d\xi}{dx} = -1$

$$\Rightarrow \frac{dk(x)}{dx} = \pi + \int_0^1 \frac{-1}{1+(\zeta-\eta)^2/4} \cdot \frac{1}{2} \frac{d\zeta}{dx}$$

change  $dy = \frac{dy}{d\eta} d\eta$  : as  $y: 0 \rightarrow 1$   
 $k: 0 \rightarrow 2\pi$   
 $\eta: +\infty \rightarrow -\infty$

$$\left. \begin{array}{l} \eta = \cot \frac{k(y)}{2} \end{array} \right\}$$

$$\Rightarrow \frac{dk(x)}{dx} = \pi + \int_{+\infty}^{-\infty} \frac{-2}{4+(\zeta-\eta)^2} \left( \frac{d\zeta}{dx} \right) \frac{dy}{d\eta} d\eta = \pi + \int_{-\infty}^{+\infty} \frac{2}{4+(\zeta-\eta)^2} \frac{dy}{d\eta} d\eta \cdot \frac{d\zeta}{dx}$$

Introduce a function  $\frac{d\zeta(x)}{dx} = -\frac{1}{g(\zeta)}$   $\frac{d\eta(y)}{dy} = -\frac{1}{g(y)}$

we have  $\frac{d\zeta(x)}{dx} = \frac{d}{dx} \left[ \cot \frac{k(x)}{2} \right] = -\frac{1}{2} \frac{1}{\sin^2 \frac{k(x)}{2}} \frac{dk(x)}{dx}$

$$= -\frac{1}{2} \left[ 1 + \cot^2 \frac{k(x)}{2} \right] \frac{dk(x)}{dx} = -\frac{1}{2} (1 + \zeta^2(x)) \frac{dk(x)}{dx}$$

$$\Rightarrow \frac{dk(x)}{dx} = \frac{2}{1+\zeta^2} \frac{1}{g(\zeta)}$$

$$\Rightarrow \frac{2}{1+\zeta^2} \frac{1}{g(\zeta)} = \pi + 2 \int_{-\infty}^{+\infty} \frac{d\eta}{4+(\zeta-\eta)^2} \quad (-) \quad g(\eta) \quad (-) \quad \frac{1}{g(\zeta)}$$

$$\Rightarrow \frac{2}{1+\zeta^2} = \pi g(\zeta) + 2 \int_{-\infty}^{+\infty} \frac{g(\eta)}{4+(\zeta-\eta)^2} d\eta$$

←  $g(\zeta)$  is rapidity spectra

This equation can be solved by Fourier transform

define  $G(u) = \int_{-\infty}^{+\infty} g(\zeta) e^{i\zeta u} d\zeta$

and  $g(\zeta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(u) e^{-i\zeta u} du$

Remember  $\int_{-\infty}^{+\infty} \frac{e^{ikx}}{1+x^2} dx = \dots \pi e^{-|k|}$

We perform Fourier transform:

$$\int_{-\infty}^{+\infty} \frac{2}{1+\zeta^2} e^{i\zeta u} d\zeta = \pi \int_{-\infty}^{+\infty} g(\zeta) e^{i\zeta u} d\zeta + 2 \int_{-\infty}^{+\infty} d\zeta \int_{-\infty}^{+\infty} d\eta \frac{g(\eta) e^{i\zeta u}}{4+(\zeta-\eta)^2} d\eta$$

$$2\pi e^{-|u|} = \pi g G(u) + \int_{-\infty}^{+\infty} d\eta g(\eta) \int_{-\infty}^{+\infty} d\zeta \frac{e^{i\frac{\zeta-\eta}{2}u}}{1+(\frac{\zeta-\eta}{2})^2}$$

$$2\pi e^{-|u|} = \pi \frac{G(u)}{G(u)} + \int_{-\infty}^{+\infty} d\eta g(\eta) \pi e^{-2|u|} e^{i\eta u} = \pi \frac{G(u)}{G(u)} + \pi e^{-2|u|} \frac{G(u)}{G(u)}$$

$$G(u) = \frac{2e^{-|u|}}{1+e^{-2|u|}} = \frac{2}{e^{|u|} + e^{-|u|}} = \boxed{\frac{1}{\cosh u} = G(u)}$$

$$\Rightarrow \frac{E}{N_J} = \frac{1}{2} \int_0^1 dx [\cos k(x) - 1] = - \int_0^1 dx \sin^2 \frac{k(x)}{2} = - \int_{-\infty}^{+\infty} d\zeta \frac{dx}{d\zeta} \frac{1}{1+\cot^2 \frac{k(x)}{2}}$$

$$= \int_{-\infty}^{+\infty} d\zeta (-) \frac{g(\zeta)}{1+\zeta^2} = (-) \int_{-\infty}^{+\infty} d\zeta \int_{-\infty}^{+\infty} \frac{du}{2\pi} \frac{e^{-i\zeta u}}{1+\zeta^2} G(u)$$

$$= - \int_{-\infty}^{+\infty} du \frac{\frac{1}{2} e^{-|u|}}{\cosh u} = - \int_0^{+\infty} du \frac{e^{-u}}{\cosh u} = - \ln 2$$

$$- \int_1^0 \frac{de^{-u}}{e^u + e^{-u}} = 2 \int_0^1 \frac{dx}{x+x^{-1}} = \int_0^1 \frac{2x dx}{x^2+1} = \ln(x^2+1) \Big|_0^1 = \ln 2$$

$$\Rightarrow \text{ground state energy } \boxed{\frac{E}{N_J} = - \ln 2}$$

A few other quantizes.

① Rapidity ( $\xi$ ) spectra  $g(\xi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(u) e^{-i\xi u} du = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\xi u}}{\cosh u} du$

The poles of  $\frac{1}{\cosh u}$  is at  $u = i(n + \frac{1}{2})\pi$ .

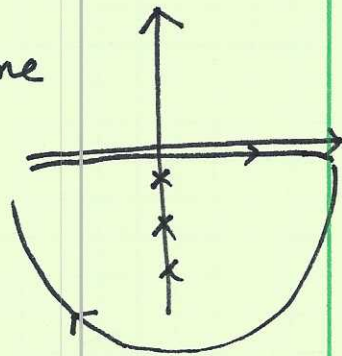
If  $\xi > 0$ , we chose counter in the lower half plane

$$\text{Res } \frac{1}{\cosh u} \Big|_{u=u_n} = \frac{1}{\sinh u} \Big|_{u=u_n} = \frac{1}{-i \sin(n + \frac{1}{2})\pi}$$

$$= (i)(-)^n$$

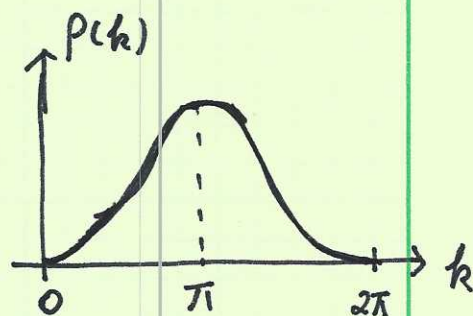
$$\Rightarrow g(\xi) = \sum_{n=0}^{+\infty} (i)(-)^n e^{-i\xi i(n + \frac{1}{2})\pi} = \sum_{n=0}^{\infty} (-)^n e^{\xi(n + \frac{1}{2})\pi}$$

$$= \frac{e^{\xi/2\pi}}{1 + e^{\xi\pi}} = \frac{1}{e^{-\xi/2\pi} + e^{\xi/2\pi}} = \boxed{\frac{1}{2 \cosh \frac{\pi\xi}{2}} = g(\xi)}$$



then  $\frac{dk(x)}{dx} = \frac{2}{1+\xi^2} \frac{1}{g(\xi)} = \frac{2}{1 + \cot^2 \frac{k}{2}} \quad 2 \cosh \frac{\pi\xi}{2} = 4 \sin^2 \frac{k}{2} \cosh \left( \frac{\pi}{2} \cot \frac{k}{2} \right)$

$$\Rightarrow \frac{dx}{dk} = \frac{1}{4 \sin^2 \frac{k}{2} \cosh \left( \frac{\pi}{2} \cot \frac{k}{2} \right)} = p(k)$$



ground state momentum



Now use another parameterization of Bethe Quantum # to recalculate the ground state energy.

$$N k_j = 2\pi m_j + \sum_{l \neq j} \phi_{jl}, \quad m_j \in \{0, 1, 2, \dots, N-1\} \text{ mod } N$$

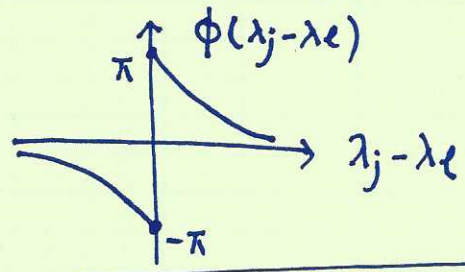
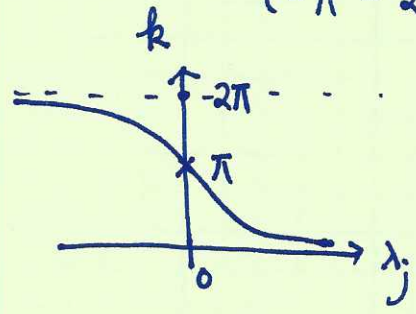
Now express  $k_j$  and  $\phi_{jl}$  using rapidities defined  $\lambda_j = \frac{1}{2} \cot \frac{k_j}{2}$

$$2\lambda_j = \cot \frac{k_j}{2} \Rightarrow \frac{k_j}{2} = \frac{\pi}{2} - \tan^{-1} 2\lambda_j \quad \text{or} \quad \boxed{k_j = \pi - 2 \tan^{-1} 2\lambda_j}$$

if take  $\tan^{-1} 2\lambda_j \in (-\frac{\pi}{2}, \frac{\pi}{2}) \Rightarrow k_j \in [0, 2\pi]$ .

$$2 \cot \frac{1}{2} \phi_{j\ell} = \cot \frac{k_j}{2} - \cot \frac{k_\ell}{2} \Rightarrow \cot \frac{1}{2} \phi_{j\ell} = \lambda_j - \lambda_\ell$$

$$\Rightarrow \phi_{j\ell} = \begin{cases} \pi - 2 \tan^{-1} (\lambda_j - \lambda_\ell) & \text{for } \text{Re}(\lambda_j - \lambda_\ell) > 0 \\ -\pi - 2 \tan^{-1} (\lambda_j - \lambda_\ell) & \text{for } \text{Re}(\lambda_j - \lambda_\ell) < 0 \end{cases}$$



$$\boxed{\phi(\lambda_j - \lambda_\ell) = \pi \text{sgn}(\lambda_j - \lambda_\ell) - 2 \tan^{-1} (\lambda_j - \lambda_\ell)}$$

plug in Bethe equation

$$N[\pi - 2 \tan^{-1} 2\lambda_j] = 2\pi m_j + \sum_{l \neq j} \pi \text{sgn}(\lambda_j - \lambda_\ell) - 2 \tan^{-1} (\lambda_j - \lambda_\ell)$$

$$\boxed{\tan^{-1} 2\lambda_j = \frac{\pi}{N} I_j + \frac{1}{N} \sum_{l \neq j} \tan^{-1} (\lambda_j - \lambda_\ell)}$$

where  $I_j = \frac{N}{2} - m_j - \frac{1}{2} \sum_{l \neq j} \text{sgn}(\lambda_j - \lambda_\ell)$ ,  $\leftarrow I_j$  takes integers or half integers

mapping  $m_j = \{1, 3, 5, \dots, N-1\}$  to the set of  $I_j$

$\lambda$  is decreasing function of  $k \Rightarrow \begin{cases} \lambda_1 > \lambda_2 > \lambda_3 > \dots \\ \Leftrightarrow k_1 < k_2 < k_3 < \dots \end{cases}$

$$\sum_{l \neq j} \text{sgn}(\frac{\lambda_l - \lambda_j}{2}) = -(j-1) + (\frac{N}{2} - j) = \frac{N}{2} - 2j + 1$$

$$\Rightarrow I_j = \frac{N}{2} - m_j - [\frac{N}{4} - j + \frac{1}{2}] = \frac{N}{4} - m_j + j - \frac{1}{2} \} \Rightarrow I_j = \frac{N}{4} - j + \frac{1}{2}$$

plug in  $m_j = 2j - 1$

$\Rightarrow I_j = \{ \frac{N}{4} - \frac{1}{2}, \frac{N}{4} - \frac{3}{2}, \dots, -\frac{N}{4} + \frac{1}{2} \}$ , we often reverse the

sequence  $\Rightarrow I_j = \{ -\frac{N}{4} + \frac{1}{2}, -\frac{N}{4} + \frac{3}{2}, \dots, \frac{N}{4} - \frac{3}{2}, \frac{N}{4} - \frac{1}{2} \}$

Let  $\chi = \frac{I_j}{N} \in [-\frac{1}{4}, \frac{1}{4}] \Rightarrow \frac{1}{N} \sum_j = \int_{-\frac{1}{4}}^{\frac{1}{4}} dx$

$$\tan^{-1} 2\lambda(x) = \pi x + \int_{-\frac{1}{4}}^{\frac{1}{4}} dy \tan^{-1}(\lambda(x) - \lambda(y))$$

define the spectra function of rapidities  $\rho(\lambda) = \frac{dx}{d\lambda}$

After reversing the direction of  $\chi$ ;  $\lambda$  increases  $\rightarrow k$  decrease  $\rightarrow \chi$  increases  
or  $\chi$  increases  $\rightarrow k$  decreases  $\rightarrow \lambda$  increases

$\Rightarrow \rho(\lambda) > 0$ . Take derivative

$$\frac{2}{1+4\lambda^2} = \pi \rho(\lambda) + \int_{-\frac{1}{4}}^{\frac{1}{4}} dy \frac{1}{1+(\lambda-\lambda(y))^2}$$

$$= \pi \rho(\lambda) + \int_{-\infty}^{+\infty} d\lambda(y) \frac{dy}{d\lambda(y)} \frac{1}{1+(\lambda-\lambda(y))^2}$$



$$\frac{2}{1+4\lambda^2} = \pi p(\lambda) + \int_{-\infty}^{+\infty} d\mu \frac{p(\mu)}{1+(\lambda-\mu)^2}$$

Again use Fourier transf:  $\int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} \frac{e^{i\omega\lambda}}{a^2+\lambda^2} = \frac{e^{-a|\omega|}}{2a} \quad (a>0)$

define  $\tilde{p}(\omega) = \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} e^{i\omega\lambda} p(\lambda)$  and  $p(\lambda) = \int_{-\infty}^{+\infty} d\omega e^{-i\omega\lambda} \tilde{p}(\omega)$

$$\int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} \frac{2}{1+4\lambda^2} e^{i\omega\lambda} = \pi \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} p(\lambda) e^{i\omega\lambda} + 2\pi \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} \int_{-\infty}^{+\infty} \frac{d\mu}{2\pi} \frac{e^{i\omega(\lambda-\mu)} e^{i\omega\mu} p(\mu)}{1+(\lambda-\mu)^2}$$

$$\frac{1}{2} e^{-\frac{1}{2}|\omega|} = \pi \tilde{p}(\omega) + 2\pi \tilde{p}(\omega) \frac{e^{-|\omega|}}{2} \Rightarrow \tilde{p}(\omega) = \frac{1}{2\pi} \frac{e^{-\frac{1}{2}|\omega|}}{1+e^{-|\omega|}}$$

$$\Rightarrow \tilde{p}(\omega) = \frac{1}{4\pi} \frac{1}{\cosh \frac{\omega}{2}}$$

$$\rightarrow p(\lambda) = \frac{dx}{d\lambda} = \frac{1}{4\pi} \int_{-\infty}^{+\infty} d\omega e^{-i\omega\lambda} \frac{1}{\cosh \frac{\omega}{2}} = \frac{1}{2 \cosh \pi\lambda}$$

$$\frac{dx}{dk} = \frac{dx}{d\lambda} \frac{d\lambda}{dk} = p(\lambda) \frac{1}{4} \frac{1}{\sin^2 \frac{k}{2}} = \frac{1}{8 \sin^2 \frac{k}{2} \cosh(\frac{\pi}{2} \cot \frac{k}{2})}$$

There's a factor of 2-difference from we obtained before. This is because the normalization  $x \in [-1/4, 1/4]$  here.

$$\begin{aligned} \frac{E}{N} &= -\frac{J}{2N} \sum_{j=1}^{N/2} \frac{1}{\lambda_j^2 + 1/4} = -\frac{J}{2} \int_{-1/4}^{1/4} \frac{dx}{\lambda^2(x) + 1/4} = -\frac{J}{2} \int_{-\infty}^{+\infty} \frac{dx}{d\lambda} \frac{1}{\lambda^2 + 1/4} \\ &= -\frac{J}{2} \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda^2 + 1/4} = -\frac{J}{2} \int_{-\infty}^{+\infty} \frac{\sec \frac{\pi}{2} \lambda}{\lambda^2 + 1/4} = -\frac{J}{2} \int_{-\infty}^{+\infty} \frac{\sec \frac{\pi}{2} y}{y^2 + 1} = -J \ln 2 \end{aligned}$$

## Two-spin excitation

(14)

Ground state Bethe quantum #s  $\{I_i\} = \{-\frac{N}{4} + \frac{1}{2}, -\frac{N}{4} + \frac{3}{2}, \dots, \frac{N}{4} - \frac{1}{2}\}$

The corresponding pattern for  $m_j$ :  $\{1, 3, 5, 7, \dots, N-1\}$ .  $\leftarrow \frac{N}{2}$  particles

For the sector of  $S_z=1$ , there're only  $\frac{N}{2}-1$  particles. The ground state

$m_j$ 's pattern:  $m_j = \{2, 4, \dots, N-2\}$ , which is more close to

center, or  $m_j = 2j$  for  $j=1, \dots, \frac{N}{2}-1$ .

Again for  $k_1 < k_2 < \dots$ , we have  $\lambda_1 > \lambda_2 > \dots$ . We have

$$\sum_{l \neq j} \text{sgn} \left( \frac{\lambda_j - \lambda_l}{2} \right) = -(j-1) + \left( \frac{N}{2} - S_z - j \right) = \frac{N}{2} - 2j - S_z + 1$$

$$I_j = \frac{N}{2} - m_j - \frac{1}{2} \sum_{l \neq j} \text{sgn} \frac{\lambda_j - \lambda_l}{2} = \frac{N}{4} - m_j + j + \frac{S_z}{2} - \frac{1}{2} \rightarrow \boxed{\frac{N}{4} - (m_j - j)}$$

for  $S_z=1$

For the ground state with  $S_z=1$ , the  $\{I_i\} = \frac{N}{4} - j$  for  $j=1, \dots, \frac{N}{2}-1$

$$\Rightarrow \boxed{\{I_i\} = \{-\frac{N}{4} + 1, \dots, \frac{N}{4} - 1\}}$$

How about the excitation spectra? We have  $1 \leq m_j \leq N-1$ . (If  $m_1=0$ ,

usually it means it comes from  $S^-$  applying on the ferro-magnetic state.

$$\Rightarrow I_1 = \frac{N}{4} + 1 - m_1 \leq \frac{N}{4}, \quad I_{\frac{N}{2}-1} = \frac{N}{4} - m_{\frac{N}{2}-1} + \left(\frac{N}{2}-1\right) \geq \frac{3}{4}N - 1 - (N-1) = -\frac{N}{4}$$

So for the range of  $1 \leq i \leq \frac{N}{2}-1$ ,  $\Rightarrow -\frac{N}{4} \leq I_i \leq \frac{N}{4}$ .

There're  $\frac{N}{2}+1$  possible values of  $I_i$ , and we need to take  $\frac{N}{2}-1$

among them, or, there are two holes to specify an excited state

in this sector.

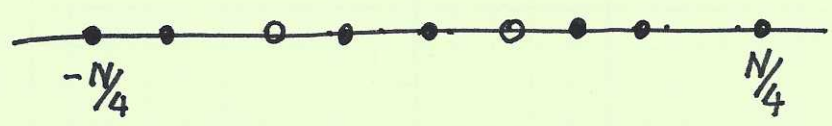
Suppose we have two jumps for  $m_j$  at the  $s$ th and  $t$ th magnon

$$m_j = \{1, 3, \dots, 2s-1, \underbrace{2s, 2s+2, \dots, 2t-2}_{1st \text{ jump}}, \underbrace{2t+1, \dots, N-1}_{2nd \text{ jump}}\}$$

$$\Rightarrow m_j = \begin{cases} 2j-1 & \text{for } i \leq s-1 \\ 2j & \text{for } s \leq i \leq t-1 \\ 2j+1 & \text{for } i \geq t \end{cases} \quad \text{then } I_i = \begin{cases} \frac{N}{4} - i + 1, & i \leq s-1 \\ \frac{N}{4} - i & s \leq i \leq t-1 \\ \frac{N}{4} - i - 1 & i \geq t \end{cases}$$

$$\text{or } I_j = \left\{ \frac{N}{4}, \dots, \frac{N}{4} - s + 2, \left\{ \frac{N}{4} - s, \dots, \frac{N}{4} - t + 1 \right\}, \frac{N}{4} - t - 1, \dots, -\frac{N}{4} \right\}.$$

the corresponding configuration



\* Now let us solve Bethe ansatz Eq

$$Nk_j = 2\pi m_j + \sum_{l \neq j} \phi_{jl}$$

$$2\lambda_j = \cot \frac{k_j}{2} \Rightarrow k_j = \pi - 2 \tan^{-1} 2\lambda_j$$

$$2 \cot \frac{1}{2} \phi_{jl} = \cot \frac{k_j}{2} - \cot \frac{k_l}{2} \Rightarrow \phi_{jl} (\lambda_j - \lambda_l) = \pi \operatorname{sgn}(\lambda_j - \lambda_l) - 2 \tan^{-1} (\lambda_j - \lambda_l)$$

$$\rightarrow \tan^{-1} 2\lambda_j = \frac{\pi}{N} I_j + \frac{1}{N} \sum_{l \neq j} \tan^{-1} (\lambda_j - \lambda_l)$$

and we reverse the sequence of  $I_j$  such that  $I_j$  is increasing

$$\text{set } x = \frac{i}{N} \quad \text{and} \quad y = \frac{I_i}{N}, \quad \text{since } I_i = -\frac{N}{4} - 1 + i + \Theta(i - S_1) + \Theta(i - S_2)$$

(  $S_1, S_2$  are related with  $s$  and  $t$ , Here we reorder them for simplicity )

$$\Rightarrow y = -\frac{1}{4} - \frac{1}{N} + x + \frac{1}{N} \Theta(x - x_1) + \frac{1}{N} \Theta(x - x_2), \quad \text{where } x_1 = \frac{S_1}{N}$$

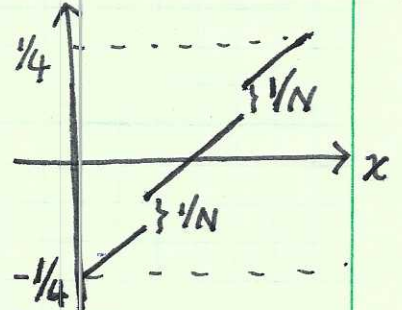
$$x_2 = \frac{S_2}{N}.$$

$$\tan^{-1} 2\lambda(x) = \pi y(x) + \int_{-1/4}^{1/4} dx' \tan^{-1}(\lambda(x) - \lambda(x'))$$

$$\frac{2}{1+4\lambda^2} = \pi \frac{dy}{dx} \frac{dx}{d\lambda} + \int_{-1/4}^{1/4} dx' \frac{1}{1+(\lambda-\lambda(x'))^2}$$

$$= \pi \frac{dx}{d\lambda} \left( 1 + \frac{1}{N} \delta(x-x_1) + \frac{1}{N} \delta(x-x_2) \right) + \int_{-\infty}^{+\infty} \frac{1}{1+(\lambda-\lambda')^2} \frac{dx'}{d\lambda}$$

$$= \pi \frac{dx}{d\lambda} + \frac{\pi}{N} \delta(\lambda - \lambda_1(x_1)) + \frac{\pi}{N} \delta(\lambda - \lambda_2(x_2)) + \int_{-\infty}^{+\infty} \frac{1}{1+(\lambda-\lambda')^2} \frac{dx'}{d\lambda}$$



$$\frac{2}{1+4\lambda^2} - \frac{\pi}{N} \delta(\lambda - \lambda_1(x_1)) - \frac{\pi}{N} \delta(\lambda - \lambda_2(x_2)) = \pi \frac{dx}{d\lambda} + \int_{-\infty}^{+\infty} \frac{1}{1+(\lambda-\lambda')^2} \frac{dx'}{d\lambda}$$

again  $\rho(\lambda) = \frac{dx}{d\lambda}$ ,  $\tilde{\rho}(\omega) = \int \frac{d\lambda}{2\pi} e^{i\omega\lambda} \rho(\lambda)$ .

$$\frac{1}{2} e^{-\frac{1}{2}|\omega|} - \frac{\pi}{N} [e^{i\omega\lambda_1} + e^{i\omega\lambda_2}] = \pi \tilde{\rho}(\omega) [1 + e^{-|\omega|}]$$

$$\Rightarrow \tilde{\rho}(\omega) = \tilde{\rho}^{(0)}(\omega) - \frac{1}{N} \frac{e^{i\omega\lambda_1} + e^{i\omega\lambda_2}}{1 + e^{-|\omega|}}, \text{ where } \tilde{\rho}^{(0)}(\omega) = \frac{1}{4\pi} \frac{1}{\cosh \frac{\omega}{2}}$$

Now calculate the excitation energy

$$E = -\frac{J}{2} \sum_{j=1}^{N/2-1} \frac{1}{\lambda_j^2 + 1/4} = -\frac{JN}{2} \int_{-1/4}^{1/4} dx \frac{1}{\lambda^2(x) + 1/4} = -\frac{JN}{2} \int_{-\infty}^{+\infty} dx \frac{dx}{d\lambda} \frac{1}{\lambda^2 + 1/4}$$

$$\Delta E = -\frac{JN}{2} \int_{-\infty}^{+\infty} d\lambda \left( -\frac{1}{N} \right) \int_{-\infty}^{+\infty} d\omega e^{-i\omega\lambda} \frac{e^{i\omega\lambda_1} + e^{i\omega\lambda_2}}{1 + e^{-|\omega|}} \frac{1}{\lambda^2 + 1/4}$$

$$\Delta E = \frac{J}{2} \int_{-\infty}^{+\infty} d\omega \frac{e^{i\omega\lambda_1} + e^{i\omega\lambda_2}}{1 + e^{-|\omega|}} \left[ \int_{-\infty}^{+\infty} d\lambda \frac{e^{-i\omega\lambda}}{\lambda^2 + 1/4} \right] e^{-\frac{1}{2}|\omega|}$$

$$\Delta E = \frac{J}{2} \int_{-\infty}^{+\infty} d\omega \frac{e^{i\omega\lambda_1} + e^{i\omega\lambda_2}}{2 \cosh \frac{\omega}{2}} = \frac{J}{2} \int_{-\infty}^{+\infty} d\omega \frac{e^{iz\omega\lambda_1} + e^{iz\omega\lambda_2}}{\cosh \omega}$$

use  $\frac{1}{2\pi} \int_{-\infty}^{+\infty} du \frac{e^{-i\beta u}}{\cosh u} = \frac{1}{2 \operatorname{ch} \frac{\pi\beta}{2}}$

$$\Rightarrow \Delta E = \frac{J}{2} \pi \left[ \frac{1}{\cosh \pi\lambda_1} + \frac{1}{\cosh \pi\lambda_2} \right]$$

It's interesting that excitations are summed  $\rightarrow$  just two holes.

But we still need to determine the values of  $\lambda_1$  and  $\lambda_2$  at the discontinuous point. At  $\lambda \rightarrow -\infty$ ,  $k \rightarrow$  largest,  $\chi \Rightarrow \frac{N/2 - 1}{N} \rightarrow 1/2$ .

$\chi(\lambda_1) - \chi(-\infty) = \int_{-\infty}^{\lambda_1} \frac{d\chi}{d\lambda} d\lambda$ , we can use the zeroth order value

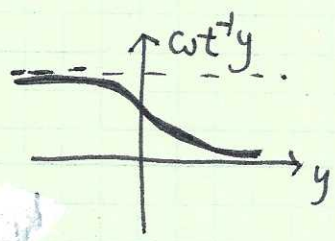
$P^0(\lambda) = \frac{1}{2 \cosh \pi\lambda} \Rightarrow \chi(\lambda_1) = \frac{1}{2} - \int_{-\infty}^{\lambda_1} \frac{d\lambda}{2 \cosh \pi\lambda}$

define  $y = \sinh \pi\lambda \Rightarrow dy = \pi \cosh \pi\lambda d\lambda$

$\Rightarrow \chi(\lambda_1) = 1/2 - \int_{-\infty}^{y_1} \frac{dy}{2\pi(1+y^2)}$  where  $y_1 = \sinh \pi\lambda_1$ .

$= 1/2 + \frac{1}{2\pi} \cot^{-1} y \Big|_{-\infty}^{y_1} = \frac{1}{2\pi} \cot^{-1} y_1 + \left[ 1/2 - \frac{1}{2\pi} \cot^{-1} y \Big|_{y \rightarrow -\infty} \right]$

we use  $\cot^{-1} y \Big|_{y \rightarrow -\infty} \rightarrow \pi$



$\Rightarrow \chi(\lambda_1) = \frac{1}{2\pi} \cot^{-1} (\sinh \pi\lambda_1)$

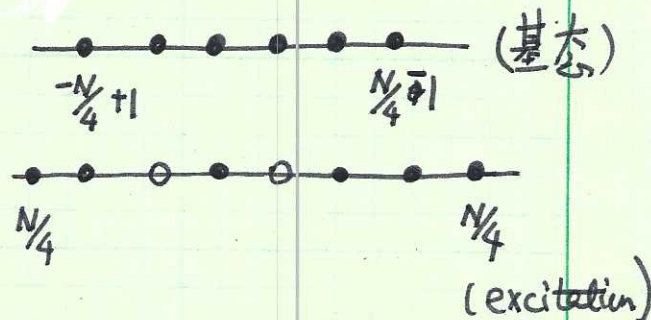
What's momentum of the system

according to the relation  $I_j = \frac{N}{4} - (m_j - j)$

$$\sum_{j=1}^{\frac{N}{2}-1} k_j = \frac{2\pi}{N} \sum_{j=1}^{\frac{N}{2}-1} m_j = \frac{2\pi}{N} \sum_{j=1}^{\frac{N}{2}-1} \left( \frac{N}{4} + j \right) - \frac{2\pi}{N} \sum_{j=1}^{\frac{N}{2}-1} I_j$$

$$\sum_{j=1}^{\frac{N}{2}-1} \left( \frac{N}{4} + j \right) = \frac{N}{4} \left( \frac{N}{2} - 1 \right) + \left( \frac{N}{2} - 1 \right) \left( \frac{N}{2} \right) \frac{1}{2}$$

$$= \frac{N}{2} \left( \frac{N}{2} - 1 \right)$$



$$\Rightarrow \sum_{j=1}^{\frac{N}{2}-1} k_j = \pi \left( \frac{N}{2} - 1 \right) - \frac{2\pi}{N} \sum_{j=1}^{\frac{N}{2}-1} I_j$$

$$\Rightarrow \Delta K = \frac{2\pi}{N} S_1 + \frac{2\pi}{N} S_2$$

$$\chi(\lambda_1) = \frac{S_1}{N} = \frac{q_1}{2\pi} \Rightarrow \frac{q_1}{2\pi} = \frac{1}{2\pi} \cot^{-1}(\sinh \pi \lambda_1)$$

$$\cot q_1 = \sinh \pi \lambda_1 \Rightarrow \frac{1}{\cosh \pi \lambda_1} = \frac{1}{\sqrt{1 + \sinh^2 \pi \lambda_1}}$$

$$= \frac{1}{\sqrt{1 + \cot^2 q_1}} = |\sin q_1|$$

Similarly calculation can be performed at the 2nd discontinuous point

$$\left\{ \begin{aligned} \chi(\lambda_2) &= \frac{S_2}{N} = \frac{q_2}{2\pi} \Rightarrow \cot q_2 = |\sin q_2| \\ \chi(\lambda_2) &= \frac{1}{2\pi} \cot^{-1}(\sinh \pi \lambda_2) \end{aligned} \right.$$

$$\Rightarrow \Delta K = q_1 + q_2, \text{ with } \Delta E = \frac{J\pi}{2} [|\sin q_1| + |\sin q_2|].$$

$$\text{with } q_1 = \frac{2\pi}{N} S_1, q_2 = \frac{2\pi}{N} S_2$$