Lect 7: Primary fields

\# A primary field is a field that corresponds to a vaccum

The primary field corresponding to the momentum p vaccum is denoted $\lim_{z\to 0} V_p(z)|0\rangle = |p\rangle$. we have $V_0(z)=1$.

We have different definitions of vaccum:

- Free boson vacuum : $\begin{cases} a_0|p\rangle = p|p\rangle \\ a_n|p\rangle = 0, \ \forall n>0 \end{cases}$

- A conformal vacuum $L_0|h\rangle = h|h\rangle$ and $L_n|h\rangle = 0$, $\forall n>0$.

  where $L_n = \frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{n-r}$.

☆ Since $L_0|p\rangle = \frac{1}{2}p^2|p\rangle$, but $L_n|p\rangle = 0$ for $\forall n>0$, a free boson vacuum is a conformal vaccum, but a conformal vacuum is not necessary a free boson vacuum.

Prove: for $n>0$, $a_r a_{n-r} = a_{n-r} a_r$, and we will not have $r<0$, and $n-r<0$. Hence we have $a_r a_{n-r}|p\rangle = 0$

$\Rightarrow L_n|p\rangle = 0$ for $n>0$.

☆ : $\partial\varphi(z)$ is conformal primary:

we have showed $\lim_{z\to 0} \partial\varphi(z)|0\rangle = a_{-1}|0\rangle$. This is not a $\underbrace{\text{free boson vacuum}}_{\text{since } a_1 a_{-1}|0\rangle = |0\rangle}$. But it is a conformal vacuum.

$L_0 |\partial\varphi\rangle = L_0 a_{-1} |0\rangle$ according to $[L_m, a_n] = -n\, a_{m+n}$

$= (\cancel{a_{-1} L_0} + [L_0, a_{-1}]) |0\rangle = a_{-1} |0\rangle = |\partial\varphi\rangle$

$\Rightarrow h = 1.$

$L_1 |\partial\varphi\rangle = L_1 a_{-1} |0\rangle = (\cancel{a_{-1} L_1} + [L_1, a_{-1}]) |0\rangle = a_0 |0\rangle = 0$

$L_n |\partial\varphi\rangle = L_n a_{-1} |0\rangle = (\cancel{a_{-1} L_n} + [L_n, a_{-1}]) |0\rangle = a\, a_{n-1} |0\rangle = 0$

for $n \geq 2$.

☆ $T(z)$ is a conformal primary iff the central charge $C = 0$.

$|T\rangle = \lim_{z \to 0} T(z) |0\rangle$, and $T(z) = \frac{1}{2} :\partial\varphi(z)\, \partial\varphi(z):$.

we had before $|T\rangle = L_{-2} |0\rangle$

$L_0 |T\rangle = L_0 L_{-2} |0\rangle = (\cancel{L_{-2} L_0} + [L_0, L_{-2}]) |0\rangle = 2 L_{-2} |0\rangle = 2|T\rangle$

$L_1 |T\rangle = L_1 L_{-2} |0\rangle = (\cancel{L_{-2} L_1} + [L_1, L_{-2}]) |0\rangle = 3 L_{-1} |0\rangle = 0$

(see Lect 6. page 1)

$L_2 |T\rangle = L_2 L_{-2} |0\rangle = (\cancel{L_{-2} L_2} + [L_2, L_{-2}]) |0\rangle = (4 L_0 + \frac{1}{2} c) |0\rangle = \frac{c}{2} |0\rangle$

☆ ~~since $L_{-1}|0\rangle = 0$ in any theory, such that $T(z)|0\rangle$~~

㊉ Primary field and OPEs.

$$|p\rangle = \lim_{w\to 0} V_p(w)|0\rangle$$

- If $V_p(w)$ is a free boson primary, $a_0|p\rangle = p|p\rangle$ and $a_n|p\rangle = 0, \forall n>0$

we expande $R\{\partial\varphi(z)V_p(w)\} = \sum_{n\in\mathbb{Z}} \psi_n(w)(z-w)^{-n-1}$.

Then $\lim_{w\to 0} R\{\partial\varphi(z)V_p(w)\}|0\rangle = \sum_{n\in\mathbb{Z}} (z\ \ )^{-n-1} \lim_{w\to 0}\psi_n(w)|0\rangle$

Since $|z|>0 \Rightarrow \partial\varphi(z)|p\rangle = \sum_{n\in\mathbb{Z}} z^{-n-1}|\psi_n\rangle$

$\partial\varphi(z) = \sum_n a_n z^{-n-1} \Rightarrow \sum_n a_n|p\rangle z^{-n-1} = \sum_{n\in\mathbb{Z}} |\psi_n\rangle z^{-n-1}$

$\Rightarrow |\psi_n\rangle = a_n|p\rangle$.

$|\psi_0\rangle = a_0|p\rangle = p|p\rangle \Rightarrow \psi_0(w) = pV_p(w)$

$|\psi_n\rangle = a_n|p\rangle = 0 \Rightarrow \psi_n(w) = 0, \forall n\geq 1$.

$$\Rightarrow \boxed{R\{\partial\varphi(z)V_p(w)\} = \frac{pV_p(w)}{z-w} + \text{regular terms}}$$

- For conformal primary, $\phi_h(w)$, $L_0|h\rangle = h|h\rangle$, $L_n|h\rangle = 0$. $\forall n>0$.

$|h\rangle = \lim_{w\to 0}\phi_h(w)|0\rangle$

expand $R\{T(z)\phi_h(w)\}$

$= \sum_{n\in\mathbb{Z}} \psi_n(w)(z-w)^{-n-2}$.

Then $\lim_{w\to 0} T(z)\phi_h(w)|0\rangle = \sum_{n\in\mathbb{Z}} z^{-n-2}\lim_{w\to 0}\psi_n(w)|0\rangle$

$T(z)|h\rangle = \sum_{n\in\mathbb{Z}} z^{-n-2}|\psi_n\rangle$

$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \Rightarrow L_n|h\rangle = |\psi_n\rangle$

Apply it to $|0\rangle$

$$\lim_{\omega \to 0} T(z)\, \phi_h(\omega)|0\rangle = \sum_{n\in\mathbb{Z}} \lim_{\omega\to 0} \psi_n(\omega)|0\rangle \cdot z^{-n-2} = \sum_{n\in\mathbb{Z}} |\psi_n\rangle z^{-n-2}$$

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \Rightarrow \sum_{n\in\mathbb{Z}} \left(L_n |h\rangle\right) z^{-n-2} = \sum_{n\in\mathbb{Z}} |\psi_n\rangle z^{-n-2}$$

$$\Rightarrow L_n|h\rangle = |\psi_n\rangle. \quad \Rightarrow \quad |\psi_0\rangle = L_0|h\rangle = h|h\rangle \Rightarrow \psi_0(\omega) = h\phi_h(\omega)$$

$$|\psi_n\rangle = L_n|h\rangle = 0 \text{ for } n \geq 1.$$

But there's an additional singular term $n=-1$.

$$|\psi_{-1}\rangle = L_{-1}|h\rangle. \longrightarrow \psi_{-1}(\omega) = \partial\phi_h(\omega).$$

We have used the following result:

If $\lim_{z\to 0} A(z)|0\rangle = |A\rangle$, then $\lim_{z\to 0} \partial A(z)|0\rangle = L_{-1}|A\rangle$.

need proof.

We have

$$R\{T(z)\,\phi_h(\omega)\} \sim \frac{h\,\phi_h(\omega)}{(z-\omega)^2} + \frac{\partial\phi_h(\omega)}{(z-\omega)}$$

Correlation functions:

Consider a physical matrix element $\langle\phi| A(z)B(w)|\psi\rangle$. Since $|\psi\rangle = \lim_{z\to 0}\psi(z)$ $|$

we only need to consider correlation functions between vacuum.

Defined $\langle 0|\underbrace{A(z)B(w)\cdots}_{\text{radically ordered}}|0\rangle \equiv \langle A(z)B(w)\cdots\rangle$, where $|0\rangle$ is interpreted at $t=-\infty$, and ~~E(z)~~ $|z|>|w|>\cdots$. $\langle 0|$ project to the true vacuum, which is intepreted at $t\to+\infty$.

$$\langle 0|a_n = (a_n^{+}|0\rangle)^{+} = (a_{-n}|0\rangle)^{+}$$

hence $\langle 0|a_n = 0$ for $n\le 0$. (note $\begin{cases} a_n|0\rangle = 0 \text{ for } n>0 \\ a_0|0\rangle = 0 \end{cases}$)

Example:

- $\langle 0|\partial\varphi(z)|0\rangle = \sum_{n\in\mathbb{Z}}\langle 0|a_n|0\rangle z^{-n-1} = 0.$

  Since $a_n|0\rangle = 0$ for $n\ge 0$, $\langle 0|a_n = 0$ for $n\le 0$.

- $\langle 0|:\partial\varphi(z)\,\partial\varphi(w):|0\rangle = \sum\langle 0|:a_r a_s:|0\rangle z^{-r-1} w^{-s-1}$

  $$:a_r a_s: = \begin{cases} a_r a_s & \text{if } r\le -1 \quad \Rightarrow \langle 0|a_s = 0 \\ a_s a_r & \text{if } r\ge 0 \quad\;\; a_s|0\rangle = 0 \end{cases}$$

  Similarly $\langle 0|T(z)|0\rangle = \frac{1}{2}\langle 0|:\partial\varphi\,\partial\varphi:|0\rangle = 0.$

- $\langle 0|R\{\partial\varphi(z)\,\partial\varphi(w)|0\rangle = \langle 0|\frac{1}{(z-w)^2} + :\partial\varphi(z)\,\partial\varphi(w):|0\rangle$

  $$= \frac{1}{(z-w)^2}$$

- $\langle 0|R\{T(z)T(w)\}|0\rangle = \dfrac{c/2}{(z-w)^4}$

- $\langle 0 | R\{\partial\varphi(z_1)\,\partial\varphi(z_2)\,\partial\varphi(z_3)\} | 0 \rangle = 0$

- $\langle 0 | R\{\partial\varphi(z_1)\,\partial\varphi(z_2)\,\partial\varphi(z_3)\,\partial\varphi(z_4)\} | 0 \rangle$

$$= \frac{1}{(z_1-z_2)^2(z_3-z_4)^2} + \frac{1}{(z_1-z_3)^2(z_2-z_4)^2} + \frac{1}{(z_1-z_4)^2(z_2-z_3)^2}$$

- Constraint on correlation function

① Let $V_{p_1}(z_1), \cdots V_{p_n}(z_n)$ be free boson primary, we have

$\langle 0 | V_{p_1}(z_1) \cdots V_{p_n}(z_n) | 0 \rangle = 0$ unless $\sum_{j=1}^{n} p_j = 0$.

Proof: $$[a_m, V_p(w)] = \oint_{|z|>|w|} \partial\varphi(z)\, V_p(w)\, z^m \frac{dz}{2\pi i} - \oint_{|z|<|w|} V_p(w)\, \partial\varphi\, z^m \frac{dz}{2\pi i}$$

$$= \oint_w R\{\partial\varphi\, V_p(w)\}\, z^m \frac{dz}{2\pi i} = \oint_w \frac{p V_p(w)}{z-w}\, z^m \frac{dz}{2\pi i}$$

$$= p\, w^m\, V_p(w)$$

hence $[a_0, V_p(w)] = p V_p(w)$. Since $\langle 0 | a_0 = 0$

we have $0 = \langle 0 | a_0 V_{p_1}(z_1) \cdots V_{p_n}(z_n) | 0 \rangle$

$$= \langle 0 | V_{p_1}(z_1)\, a_0 \cdots V_{p_n}(z_n) | 0 \rangle + \langle 0 | [a_0, V_{p_1}(z_1)] \cdots V_{p_n}(z_n) | 0 \rangle$$

$$= \cdots \langle 0 | V_{p_1}(z_1) \cdots V_{p_n}(z_n)\, a_0 | 0 \rangle + \sum_{j=1}^{n} \langle 0 | V_{p_1}(z_1) \cdots [a_0, V_{p_j}(z_j)] \cdots V_{p_n}(z_n) | 0 \rangle$$

$$= 0 + \left(\sum_{j=1}^{n} p_j\right) \langle 0 | V_{p_1}(z_1) \cdots V_{p_n}(z_n) | 0 \rangle$$

$\Rightarrow$ result of momentum conservation.

For conformal primaries $\phi_h(z)$, we have

$$[L_M, \phi_h(\omega)] = h(M+1)\,\omega^M \phi_h(\omega) + \omega^{M+1}\partial\phi_h(\omega)$$

Proof: $T(z) = \sum_{M\in\mathbb{Z}} L_M z^{-M-2} \Rightarrow L_M = \oint T(z) z^{M+1}\frac{dz}{2\pi i}$

$$[L_M, \phi_h(\omega)] = \oint_{|z|>|\omega|} T(z)\phi_h(\omega) z^{M+1}\frac{dz}{2\pi i} - \oint_{|z|<|\omega|} \phi_h(\omega) T(z) z^{M+1}\frac{dz}{2\pi i}$$

$$= \oint_\omega R\{T(z)\phi_h(\omega)\} z^{M+1}\frac{dz}{2\pi i}$$

$$= \oint_\omega \left[\frac{h\phi_h(\omega)}{(z-\omega)^2} + \frac{\partial\phi_h(\omega)}{(z-\omega)}\right] z^{M+1}\frac{dz}{2\pi i}$$

$$= h(M+1)\,\omega^M\phi_h(\omega) + \omega^{M+1}\partial\phi_h(\omega)$$

Then we have $\sum_{j=1}^{n}\left[h_j(m+1)z_j^m + z_j^{m+1}\partial_j\right]\cdot\langle 0|\phi_{h_1}(z_1)\cdots\phi_{h_n}(z_n)|0\rangle = 0$

$(m=0,\pm1)$

Proof: For $m = 0, \pm 1$, $\begin{cases} L_0|0\rangle = 0 \\ L_{-1}|0\rangle = 0 \\ L_n|0\rangle = 0 \text{ for } n\geq 0 \end{cases}$ $\Rightarrow L_0|0\rangle = L_\pm|0\rangle = 0$

$\Rightarrow \langle 0|L_m = 0$ for $m=0,\pm1$

$$0 = \langle 0|L_m\phi_{h_1}(z_1)\cdots\phi_{h_n}(z_n)|0\rangle$$

$$= \langle 0|\phi_{h_1}(z_1)L_m\phi_{h_2}(z_2)\cdots|0\rangle + \langle 0|[L_m\,\phi_{h_1}(z_1)]\,\phi_h(z_2)\cdots\rangle$$

$$= \langle 0|\phi_h(z_1)\cdots\phi_{h_n}(z_n)L_m|0\rangle + \sum_{j=1}^{n}\langle 0|\phi_{h_1}(z_1)[L_m\phi_{h_j}(z_j)]\cdots\phi_{h_n}(z_n)\rangle$$

$$= \sum_{j=1}^{n}\left[h_j(m+1)z_j^m + z_j^{m+1}\partial_j\right]\langle 0|\phi_{h_1}(z_1)\cdots\phi_{h_n}(z_n)|0\rangle$$

$\Rightarrow$

$m=1$ $$\sum_{j=1}^{n} \partial_j \langle 0| \phi_{h_1}(z_1) \cdots \phi_{h_n}(z_n)|0\rangle = 0$$

$m=0$ $$\sum_{j=1}^{n} (z_j \partial_j + h_j) \langle 0| \phi_{h_1}(z_1) \cdots \phi_{h_n}(z_n)|0\rangle = 0$$

$m=-1$ $$\sum_{j=1}^{n} (z_j^2 \partial_j + 2h_j z_j) \langle 0| \phi_{h_1}(z_1) \cdots \phi_{h_n}(z_n)|0\rangle = 0$$

From these sets of equations, we can derive the same scaling forms of correlation functions.