Lect 9: Hubbard model: Lieb-Wu solution

The BA equation for the continuous model can be straightforwardly generalized to the lattice Hubbard model by the substitution

$$k_i \to \sin k_i, \quad c \to U/2 \leftarrow t \text{ is set to } 1.$$

Then

$$e^{ik_jL} = \prod_{\alpha=1}^{M} \frac{k_j - \Lambda_\alpha + ic/2}{k_j - \Lambda_\alpha - ic/2} \to$$

$$e^{ik_jL} = \prod_{\alpha=1}^{M} \frac{\sin k_j - \Lambda_\alpha + iU/4}{\sin k_j - \Lambda_\alpha - iU/4} \quad ①$$

$$\prod_{j=1}^{N} \frac{k_j - \Lambda_\alpha - ic/2}{k_j - \Lambda_\alpha + ic/2} = -\prod_{\beta=1}^{M} \frac{\Lambda_\beta - \Lambda_\alpha - ic}{\Lambda_\beta - \Lambda_\alpha + ic} \to$$

$$\prod_{j=1}^{N} \frac{\sin k_j - \Lambda_\alpha - iU/4}{\sin k_j - \Lambda_\alpha + iU/4} = -\prod_{\beta=1}^{M} \frac{\Lambda_\beta - \Lambda_\alpha - iU/2}{\Lambda_\beta - \Lambda_\alpha + iU/2} \quad ②$$

define $\theta(x) = -2\tan^{-1}\left(\frac{2x}{U}\right)$, Eq ① $\to$

$$k_jL = 2\pi \cdot I_j + \sum_{\alpha=1}^{M} \theta(2\sin k_j - 2\Lambda_\alpha) \quad (a) \quad j = 1, 2, \cdots N$$

N: # of total particles

Eq② $\to$ $$\sum_{j=1}^{N} \theta(2\sin k_j - 2\Lambda_\alpha) = 2\pi J_\alpha + \sum_{\beta=1}^{M} \theta(\Lambda_\beta - \Lambda_\alpha) \quad (b) \quad \alpha = 1, \cdots M$$

M: # of spin down

or

where $I_j = \begin{cases} \text{integer} & M = \text{even} \\ \text{half integer} & M = \text{odd} \end{cases}$

$J_\alpha = \begin{cases} \text{integer} & N-M = \text{odd} \\ \text{half integer} & N-M = \text{even} \end{cases}$

In the ground state, I_j and J_α symmetrically distribute on both sides of zero. $I_{j+1} - I_j = 1$, $J_{\alpha+1} - J_\alpha = 1$.

Set $L \to \infty$, $N \to \infty$, $M \to \infty$, and N/L, M/L fixed.

Again set $L\,\rho(k)\,dk = \#$ of k_j from $k \to k + dk$

$L\,\sigma(\Lambda)\,d\Lambda = \#$ of Λ_α from $\Lambda \to \Lambda + d\Lambda$

define $f = I/L$, $g = J/L$ $\Rightarrow$ $df/dk = \rho(k)$, $dg/d\Lambda = \sigma(\Lambda)$

$$d\theta/dx = -2 \cdot \frac{u/2}{(u/2)^2 + x^2} = -\frac{4u}{u^2 + 4x^2}$$

Eq (a) $\Rightarrow$ $$k = 2\pi f + \int_{-B}^{B} \theta(2\sin k - 2\Lambda)\,\sigma(\Lambda)\,d\Lambda$$

Eq (b) $\Rightarrow$

$$-\int_{-Q}^{Q} \theta(2\Lambda - 2\sin k)\,\rho(k)\,dk = 2\pi g - \int_{-B}^{B} \theta(\Lambda - \Lambda')\,\sigma(\Lambda')\,d\Lambda'$$

$$1 = 2\pi\, df/dk + \int_{-B}^{B} \frac{-4u}{u^2 + 4(2\sin k - 2\Lambda)^2} \cdot 2\cos k\,\sigma(\Lambda)\,d\Lambda$$

$\rightarrow$

$$2\pi\rho(k) = 1 + \cos k \int_{-B}^{B} \frac{8u}{u^2 + 16(\sin k - \Lambda)^2}\,\sigma(\Lambda)\,d\Lambda \quad (*)$$

$\rightarrow$ $$-\int_{-Q}^{Q} \frac{-4u}{u^2 + 16(\Lambda - \sin k)^2} \cdot 2\, \rho(k)\,dk = 2\pi\, dg/d\Lambda - \int_{-B}^{B} \frac{-4u}{u^2 + 4(\Lambda - \Lambda')^2}\,\sigma(\Lambda')\,d\Lambda'$$

$$\int_{-Q}^{Q} \frac{8u}{u^2 + 16(\Lambda - \sin k)^2}\,\rho(k)\,dk = 2\pi\sigma(\Lambda) + \int_{-B}^{B} \frac{4u}{u^2 + 4(\Lambda - \Lambda')^2}\,\sigma(\Lambda')\,d\Lambda' \quad (**)$$

Recap BA equation

$$(*)\ \rho(k) = \frac{d(I/L)}{dk} = \frac{1}{2\pi} + \frac{4}{\pi}\cos k \int_{-B}^{B} a(4\sin k - 4\Lambda)\,\sigma(\Lambda)\,d\Lambda$$

$$(**)\ \sigma(\Lambda) = \frac{d(J/L)}{dk} = \frac{4}{\pi}\int_{-Q}^{Q} a(4\sin k - 4\Lambda)\,\rho(k)\,dk - \frac{2}{\pi}\int_{-B}^{B} a(2\Lambda - 2\Lambda')\,\sigma(\Lambda')\,d\Lambda'$$

where $a(x) = \dfrac{u}{u^2 + x^2}$,

and $\int_{-Q}^{Q} dk\,\rho(k) = N/L$, $\int_{-B}^{B} d\Lambda\,\sigma(\Lambda) = \dfrac{N_\downarrow}{L}$.

Formula

$$\int_{-\infty}^{+\infty} \frac{e^{-i\omega x}\,dx}{a^2 + x^2} = \frac{\pi}{a} e^{-a|\omega|} \quad \text{for } a > 0,$$

$$\int_{-\pi}^{\pi} \frac{dk}{2\pi}\, e^{-i\omega\sin k} = J_0(\omega), \quad \int_{-\pi}^{\pi} dk/2\pi\, \cos^2 k\, \cos(\omega \sin k) = \frac{J_1(\omega)}{\omega}$$

$$\int_{-\infty}^{+\infty} d\Lambda\, e^{-i\omega\Lambda}\, a(4\sin k - 4\Lambda) = \frac{\pi}{4} e^{-u|\omega|/4}\, e^{-i\omega \sin k}$$

$$\int_{-\infty}^{+\infty} d\Lambda\, e^{-i\omega\Lambda}\, a(2\Lambda - 2\Lambda') = \frac{\pi}{2} e^{-u|\omega|/2}\, e^{-i\omega\Lambda'}$$

$$\frac{E}{L} = -2\int_{-Q}^{Q} \rho(k)\cos k\, dk$$

(*) Solution at half-filling

$N = L, \quad N_\uparrow = L/2, \quad M = N_\downarrow = L/2$

It can be proved that as $L \to \infty$, the integral boundaries $Q=\pi$, $B=\infty$ at half-filling. And the normalization condition is $\int_{-\pi}^{\pi} \rho(k) dk = 1$ and $\int_{-\infty}^{+\infty} \sigma(\Lambda) d\Lambda = 1/2$.

Define Fourier transform:

$$\sigma(\Lambda) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \tilde{\sigma}(\omega) e^{i\omega\Lambda}, \quad \tilde{\sigma}(\omega) = \int_{-\infty}^{+\infty} \sigma(\Lambda) e^{-i\omega\Lambda} d\Lambda$$

1. Do Fourier transformation $\int_{-\infty}^{+\infty} d\Lambda \, e^{-i\omega\Lambda}$ to Eq(**).

$$\tilde{\sigma}(\omega) = \frac{4}{\pi} \cdot \frac{\pi}{4} \int_{-\pi}^{\pi} e^{-u|\omega|/4} e^{-i\omega \sin k} \rho(k) dk - \frac{2}{\pi} \cdot \frac{\pi}{2} \int_{-\infty}^{+\infty} e^{-\frac{u|\omega|}{2}} e^{-i\omega\Lambda'} \sigma(\Lambda') d\Lambda'$$

$$\tilde{\sigma}(\omega)\left[1 + e^{-u|\omega|/2}\right] = e^{-u|\omega|/4} \int_{-\pi}^{\pi} dk \, \rho(k) e^{-i\omega \sin k}$$

$$\boxed{\tilde{\sigma}(\omega) = \frac{1}{2} \operatorname{sech} \frac{u\omega}{4} \int_{-\pi}^{\pi} dk \, \rho(k) e^{-i\omega \sin k} = \frac{1}{2} \operatorname{sech} \frac{u\omega}{4} J_0(\omega)}$$

see below

2. Then apply $\int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-i\omega \sin k}$ to Eq(*)

$$\int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-i\omega \sin k} \rho(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-i\omega \sin k} + \frac{4}{\pi} \int_{-\infty}^{+\infty} d\Lambda \, \sigma(\Lambda) \int_{-\pi}^{\pi} \frac{dk}{2\pi} \cos k \, a(4\sin k - 4\Lambda) e^{-i\omega \sin k}$$

$$\int_{-\pi}^{\pi} d\sin k \, e^{-i\omega \sin k} a(4\sin k - 4\Lambda) = 0$$

$$\boxed{\int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-i\omega \sin k} \rho(k) = J_0(\omega)}$$

$\tilde{\sigma}(\omega=0) = \int_{-\infty}^{\infty} \sigma(\Lambda)\, d\Lambda = 1/2$ — consistent with the normalization.

$$\sigma(\Lambda) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{i\Lambda\omega}\, \tilde{\sigma}(\omega) = \int_{-\infty}^{+\infty} \frac{d\omega}{4\pi}\, e^{i\Lambda\omega}\, J_0(\omega)\, \mathrm{sech}\frac{u\omega}{4}$$

plug in $\sigma(\Lambda)$ into Eq (*)

$$\rho(k) = \frac{1}{2\pi} + \frac{4}{\pi}\cos k \int_{-\infty}^{+\infty} d\Lambda\, e^{i\Lambda\omega}\, a(4\sin k - 4\Lambda) \int_{-\infty}^{+\infty} \frac{d\omega}{4\pi}\, J_0(\omega)\, \mathrm{sech}\frac{u\omega}{4}$$

$$= \frac{1}{2\pi} + \frac{4}{\pi}\cos k \cdot \frac{\pi}{4} \int_{-\infty}^{+\infty} \frac{d\omega}{4\pi}\, J_0(\omega)\, \mathrm{sech}\frac{u\omega}{4}\, e^{-u|\omega|/4}\, e^{i\omega\sin k}$$

← even on ω.

$$\rho(k) = \frac{1}{2\pi} + \frac{\cos k}{\pi} \int_0^{\infty} d\omega\, \frac{J_0(\omega)\cos(\omega \sin k)}{1 + e^{u\omega/2}}$$

$$\int_{-\pi}^{\pi} \rho(k)\, dk = 1 + \int_0^{\infty} d\omega\, \frac{J_0(\omega)}{1+e^{u\omega/2}} \int_{-\pi}^{\pi} \frac{dk}{\pi} \cos k \cdot \cos(\omega \sin k)$$

↘ $\int_{-\pi}^{\pi} d\sin k\, \cos(\omega \sin k) = 0$

$$\int_{-\pi}^{\pi} \rho(k)\, dk = 1$$

← consistent with the normalization.

plug in

$$\frac{E}{L} = -2\int_{-Q}^{Q} \rho(k)\cos k\, dk = -2\int_{-\pi}^{\pi} dk\, \frac{\cos^2 k}{\pi} \int_0^{\infty} \frac{J_0(\omega)\cos(\omega \sin k)}{1+e^{u\omega/2}}\, d\omega$$

$$= -2\int_0^{\infty} d\omega\, \frac{J_0(\omega)}{1+e^{u\omega/2}} \int_{-\pi}^{\pi} \frac{dk}{\pi} \cos^2 k\, \cos(\omega \sin k)$$

$$\frac{E}{L} = -4\int_0^{\infty} d\omega\, \frac{J_0(\omega) J_1(\omega)}{\omega(1+e^{u\omega/2})}$$

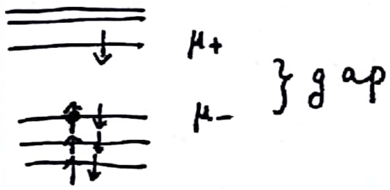
(*) Gap

Definition: $\mu_+ = E(N_\uparrow, N_\downarrow + 1) - E(N_\uparrow, N_\downarrow)$

$\mu_- = E(N_\uparrow, N_\downarrow) - E(N_\uparrow, N_\downarrow - 1)$, where $N_{\uparrow,\downarrow}$ are particle #'s of spin $\uparrow, \downarrow$.

SU(2) symmetry, only change $N_\downarrow$

if $\mu_+ - \mu_- = O(1/L)$, then it's a gapless metal



if $\mu_+ - \mu_- = \Delta \to$ finite

insulator

particle-hole symmetry:

Hubbard model when expressed as

$$H' = -t \sum_{\langle ij \rangle} (C^\dagger_{i\sigma} C_{j\sigma} + h.c) + U \sum_i (n_{i\uparrow} - 1/2)(n_{i\downarrow} - 1/2)$$

$$= H - \frac{1}{2}(N_\uparrow + N_\downarrow) U + \text{const}$$

H' has a particle-hole symmetry, that under $n_{i\sigma} \to 1 - n_{i\sigma}$

or $C_{i,\sigma} \to (-)^i C_{i\sigma}$, we have $H' \to H'$.

Hence $E'(N_\uparrow, N_\downarrow) = E'(L - N_\uparrow, L - N_\downarrow)$

$$E(N_\uparrow, N_\downarrow) = E(N'_\uparrow, N'_\downarrow) + \frac{1}{2}(N_\uparrow + N_\downarrow) U$$

$$= E'(L - N_\uparrow, L - N_\downarrow) + 1/2 (N_\uparrow + N_\downarrow) U$$

$$= E(L - N_\uparrow, L - N_\downarrow) - \frac{1}{2}(L - N_\uparrow + L - N_\downarrow) U + 1/2 (N_\uparrow + N_\downarrow) U$$

$$= E(L - N_\uparrow, L - N_\downarrow) - (L - N_\uparrow - N_\downarrow) U$$

$$E(N_\uparrow = L/2, N_\downarrow = L/2 - 1) = E(N_\uparrow = L/2, N_\downarrow = L/2 + 1) - (L - L + 1) U$$

$$= E(N_\uparrow = L/2, N_\downarrow = L/2 + 1) - U \Rightarrow \boxed{\mu_+ + \mu_- = U} \text{ when } N_\uparrow = N_\downarrow = L/2.$$

Define the spectral density function ρ_0 for the case $N=L$

and ρ for the case of $N_\uparrow = L/2 - 1$, $N_\downarrow = \frac{L}{2} - 1$. We should have ← spin up and down symmetrically

$$\int_{-Q=-\pi}^{Q=\pi} \rho_0(k)\, dk = 1, \quad \int_{-Q^-}^{Q^-} \rho(k)\, dk = 1 - 2/L.$$

when $N_\downarrow$ changes by one, I_j's pattern changes for integers ⟷ half integers.

i.e. # of I's increases by one. we express

$$\rho_0 - \rho = \frac{1}{L}\left[\delta(k-\pi) + \delta(k+\pi)\right] + \delta\rho(k) \leftarrow \quad |k| \le Q^- < Q$$

↗ Count the ~~extra~~ two particles

then

$$\mu_- = \frac{1}{2}\left[E(N=L) - E(N=L-2)\right] = -\frac{2L}{2}\left[\int_{-Q}^{Q} \cos k\, \rho_0(k)\, dk - \int_{-Q^-}^{Q^-} \cos k \;\; \rho(k)\, dk\right]$$

$$\mu_- = -2\cos Q - L \int_{-Q^-}^{Q^-} \cos k\; \delta\rho\; dk$$

(*)
$$\rho_0 = \frac{1}{2\pi} + \frac{4}{\pi}\cos k \int_{-B}^{B} a(4\sin k - 4\Lambda)\, \sigma_0(\Lambda)\, d\Lambda$$

$$\rho(k) = \frac{1}{2\pi} + \frac{4}{\pi}\cos k \int_{-B}^{B} a(4\sin k - 4\Lambda)\, \sigma(\Lambda)\, d\Lambda$$

$$\Rightarrow \delta\rho(k) + 1/L\left[\delta(k-Q) + \delta(k+Q)\right] = \frac{4}{\pi}\cos k \int_{-B}^{B} a(4\sin k - 4\Lambda)\, \delta\sigma(\Lambda)\, d\Lambda$$

where $\delta\sigma(\Lambda) = \sigma_0(\Lambda) - \sigma(\Lambda)$ (***)

(**)

$$\sigma_0(\Lambda) = \frac{d(J/L)}{dk} = \frac{4}{\pi}\int_{-Q}^{Q} a(4\sin k - 4\Lambda)\rho_0(k)dk - \frac{2}{\pi}\int_{-B}^{B} a(2\Lambda - 2\Lambda')\sigma_0(\Lambda')d\Lambda'$$

$$\sigma(\Lambda) = \frac{4}{\pi}\int_{-Q^-}^{Q^-} a(4\sin k - 4\Lambda)\rho(k)dk - \frac{2}{\pi}\int_{-B}^{B} a(2\Lambda - 2\Lambda')\sigma(\Lambda')d\Lambda'$$

$$\rightarrow \delta\sigma(\Lambda) = \frac{4}{\pi}\cdot\frac{2}{L}a(4\Lambda) + \frac{4}{\pi}\int_{-Q^-}^{Q^-} a(4\sin k - 4\Lambda)\delta\rho(k)dk - \frac{2}{\pi}\int_{-\infty}^{\infty} a(2\Lambda - 2\Lambda')\delta\sigma(\Lambda')d\Lambda' \quad (****)$$

Apply Fourier transform $\int_{-\infty}^{\infty} d\Lambda\, e^{-i\Lambda\omega}$ to Eq(****)

$$\delta\tilde{\sigma}(\omega) = \frac{8}{\pi L}\cdot\frac{\pi}{4}e^{-u|\omega|/4} + \frac{4}{\pi}\int_{-Q^-}^{Q^-} dk\,\delta\rho(k)\cdot\frac{\pi}{4}e^{-u|\omega|/4}e^{-i\omega\sin k} - \frac{2}{\pi}\cdot\frac{\pi}{2}\int_{-\infty}^{\infty} e^{-u|\omega|/2}e^{-i\omega\Lambda'}\delta\sigma(\Lambda')d\Lambda'$$

$$\delta\tilde{\sigma}(\omega) = \frac{2}{L}e^{-u|\omega|/4} + \int_{-Q^-}^{Q^-} dk\,\delta\rho(k)e^{-i\omega\sin k}e^{-u|\omega|/4} - \delta\tilde{\sigma}(\omega)e^{-u|\omega|/2}$$

$$\delta\tilde{\sigma}(\omega)\left[e^{-u|\omega|/4} + e^{u|\omega|/4}\right] = \frac{2}{L} + \underbrace{\int_{-Q^-}^{Q^-} dk\,\delta\rho(k)e^{-i\omega\sin k}}_{\to 0 \text{ (see below)}}$$

Based on (**) since $\delta(k\pm Q)$ is out of $[-Q^-, Q^-]$, we plug in Eq(***) without the δ-function.

$$\frac{4}{\pi}\int_{-\pi}^{\pi} dk\int_{-\infty}^{\infty}\underbrace{\cos k}_{d\Lambda}\, a(4\sin k - 4\Lambda)\delta\sigma(\Lambda)e^{-i\omega\sin k} - \frac{4}{\pi}\underbrace{\int_{-\infty}^{+\infty} d\Lambda\int_{-\pi}^{\pi} d\sin k\, a(4\sin k - 4\Lambda)e^{-i\omega\sin k}}_{\delta\sigma(\Lambda)}$$

$$= 0$$

$$\delta\tilde{\sigma}(\omega)\, 2\, ch\, u\omega/4 = 2/L$$

$$\boxed{\delta\tilde{\sigma}(\omega) = \frac{1}{L}\,\text{sech}\,\frac{u\omega}{4}}$$

$$\Rightarrow \int_{-\infty}^{+\infty}\delta\sigma(\Lambda)\,d\Lambda = \delta\tilde{\sigma}(0) = 1/L$$

consistent with the normalization

$$\delta\sigma(\Lambda) = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\delta\tilde{\sigma}(\omega)\,e^{i\omega\Lambda} = \frac{1}{2\pi L}\int_{-\infty}^{+\infty}d\omega\ \text{sech}\,\frac{u\omega}{4}\ e^{i\omega\Lambda}$$

plug in Eq (***)

$$\delta\rho(k) + \frac{1}{L}\left[\delta(k+\pi)+\delta(k-\pi)\right] = \frac{\cos k}{\pi L}\frac{4}{}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\ \text{sech}\frac{u\omega}{4}\int_{-\infty}^{+\infty}d\Lambda\ a(4\sin k-4\Lambda)\,e^{-i\omega\Lambda}$$

$$= \frac{\cos k}{L}\cdot\frac{4}{\pi}\cdot\frac{\pi}{4}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\ \text{sech}\frac{u\omega}{4}\ e^{-u|\omega|/4}\ e^{-i\omega\sin k}$$

$$= \frac{\cos k}{L}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\ \frac{2e^{-i\omega\sin k}}{1+e^{u|\omega|/2}}$$

$$= \frac{2\cos k}{L}\int_{0}^{+\infty}\frac{d\omega}{\pi}\ \frac{\cos(\omega\sin k)}{1+e^{u\omega/2}}$$ ← plug to the expression of μ_- on page ⑦

$$\Rightarrow\ \mu_- = 2 - \frac{2L}{L}\int_{-\pi^-}^{\pi^-}dk\,\cos^2 k\int_0^{+\infty}\frac{d\omega}{\pi}\ \frac{\cos(\omega\sin k)}{1+e^{u\omega/2}}$$ ← δ-function no contribution since it's outside the integrand.

$$= 2 - \int_0^{+\infty}d\omega\left[\int_{-\pi}^{\pi}\frac{dk}{2\pi}\cos^2 k\,\cos(\omega\sin k)\right]\frac{4}{1+e^{u\omega/2}}$$

$$\mu_- = 2 - 4\int_0^{\infty}\frac{J_1(\omega)\,d\omega}{\omega(1+e^{u\omega/2})}$$

$$\boxed{\mu_+ - \mu_- = u - 2\mu_- = u - 4 + 8\int_0^{\infty}\frac{J_1(\omega)\,d\omega}{\omega(1+e^{u\omega/2})}}$$

(*) Mott gap at $U>0$

$$\Delta = \mu_+ - \mu_- = U - 4 + 8\int_0^\infty \frac{J_1(\omega)\,d\omega}{\omega(1+e^{U\omega/2})}$$

$$J_1(\omega) = \sum_{m=0}^{\infty} \frac{(-)^m (\omega/2)^{2m+1}}{m!(m+1)!} = \frac{\omega}{2} - \frac{\omega^3}{16} + \frac{\omega^5}{384} + \cdots$$

$$\int_0^\infty \frac{\omega^n d\omega}{1+e^{U\omega/2}} = \left(\frac{2}{U}\right)^{2n+1}\int_0^\infty \frac{x^n dx}{1+e^x} = \left(\frac{2}{U}\right)^{2n+1} (1-2^{-n})\, n!\, \zeta(1+n)$$

where Zeta function $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$

$$\int_0^\infty \frac{x^s}{e^x-1}dx = \Gamma(s+1)\zeta(s+1)$$

$$\int_0^\infty \frac{x^s dx}{e^x+1} = (1-2^{-s})\Gamma(s+1)\zeta(1+s)$$

$$\int_0^\infty \frac{d\omega}{1+e^{U\omega/2}} = \frac{2}{U}\ln 2$$

$$\Rightarrow \int_0^\infty \frac{J_1(\omega)\,d\omega}{\omega(1+e^{U\omega/2})} = \sum_{m=0}^{\infty} (-)^m \frac{2^{-2m-1}}{m!(m+1)!}\int_0^\infty \frac{\omega^{2m}d\omega}{1+e^{U\omega/2}}$$

$$= \sum_{m=0}^{\infty} (-)^m \frac{1}{(m+1)!}\left(\frac{1}{U}\right)^{2m+1}(1-2^{-m})\,\zeta(1+n)$$

by comparing Taylor series

$$\int_0^\infty \frac{J_1(\omega)\,d\omega}{\omega(1+e^{U\omega/2})} = \sum_{n=1}^{\infty} (-)^{n+1}\left(\sqrt{1+\frac{n^2U^2}{4}} - \frac{nU}{2}\right)$$

$$\Rightarrow \boxed{\Delta = \mu_+ - \mu_- = U - 4 - 8\sum_{n=1}^{\infty}(-)^n\left[\sqrt{1+\frac{n^2U^2}{4}} - \frac{nU}{2}\right]}$$

$$\mu_- = 2 - 4\sum_{n=1}^{\infty}(-)^n\left[\left(1+\frac{n^2U^2}{4}\right)^{1/2} - \frac{nU}{2}\right]$$

It can be proved that

$$\boxed{\Delta = \frac{16}{U}\int_1^\infty dy \frac{\sqrt{y^2-1}}{\sinh\frac{2\pi y}{U}} > 0, \text{ for any } U>0}$$

① Strong coupling limit $u \to \infty$

$$\left(1+\frac{n^2u^2}{4}\right)^{1/2} = \frac{nu}{2}\left(1+\left(\frac{2}{nu}\right)^2\right)^{1/2} \simeq \frac{nu}{2}\left[1+\frac{2}{(nu)^2}\right] = \frac{nu}{2}+\frac{1}{nu}$$

$$\Rightarrow \Delta = u-4-8/u \sum_{n=1}^{\infty} (-)^n/n = u-4+\frac{8\ln 2}{u}+O\left(\frac{1}{u^2}\right)$$

$$\sim u-4t+(8\ln 2)t^2/u + O(1/u^2)$$ — linear to u

② weak coupling $u \to 0$

$$\Delta \propto \frac{16}{u} e^{-2\pi/u} \int_0^\infty dx \frac{\sqrt{2x}\cdot 2}{e^{2\pi x/u}} \propto \frac{t^2}{u} e^{-2\pi t/u}$$

↓ exponetially small!