Lect 11

Motion in the magnetic field — Landau levels

$$H^{LL}_{2D} = \frac{(P - \frac{e}{c}A)^2}{2m} \qquad \vec{A} = \frac{B}{2}\hat{z}\times\vec{r} = \frac{B}{2}(-y, x)$$

cyclotron radius $l_B = \sqrt{\frac{\hbar c}{|eB|}} = \frac{257\,\mathring{A}}{\sqrt{B/\mathrm{Tesla}}}$

$$H^{LL}_{2D} = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 r^2 - \frac{eB}{2mc}\hat{z}\cdot(\vec{r}\times\vec{p}), \qquad \omega_0 = \frac{|eB|}{2mc}$$

$$\boxed{H_{2D} = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 r^2 \mp \omega_0 L_z},$$ where '$\mp$' apply for $\vec{B} \parallel \pm\hat{z}$ respectively.

2D Landau level Hamiltonian is nothing but 2D Harmonic oscillator plus orbital Zeeman term. The $\omega_0 L_z$ term commutes with $\frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 r^2$, thus the 2D Landau level wavefunctions are nothing but 2D harmonic oscilator wavefunctions. But we know 2D harmonic oscillaters do not have very exciting properties, but the orbital Zeeman term reorganizes the spectra, and leads to non-trival topology.

Let us first look at the spectra and wavefunctions of 2D harmonic oscillater

$$H_{harm} = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 r^2$$

$$\left[-\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)+\frac{1}{2}m\omega_0^2r^2\right]\psi = E\psi$$

Seperation of variables $\psi(r,\varphi) = R(r)\,e^{im'\varphi}$

define length unit $l = \sqrt{\frac{\hbar}{m\omega_0}}$

$$\Rightarrow \left[l^2\frac{d^2}{dr^2}+\frac{l^2}{r}\frac{d}{dr}-\frac{m'^2}{r^2}l^2+\frac{2E}{\hbar\omega_0}-\frac{r^2}{l^2}\right]R(r)=0$$

as $r\to 0$, $\Rightarrow \left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m'^2}{r^2}\right]R(r)=0$

$$\Rightarrow R(r)\sim r^{|m|'}$$

as $r\to\infty$ $\Rightarrow \left[l^2\frac{d^2}{dr^2}-\frac{r^2}{l^2}\right]R(r)=0 \Rightarrow R(r)\sim e^{-\frac{r^2}{2l^2}}$

$\Rightarrow$ we can try $R(r) = r^{|m|'}e^{-\frac{r^2}{2l^2}}u(r) \Rightarrow u(r)$ satisfies

$$l^2\frac{d^2u}{dr^2}+\left[\frac{2|m|'+1}{r/l}-\frac{2r}{l}\right]l\frac{du}{dr}+\left[\frac{2E}{\hbar\omega_0}-2(|m|'+1)\right]u=0$$

define $\xi = r^2/l^2 \Rightarrow$

$$\xi\frac{d^2u}{d\xi^2}+\left[|m|'+1-\xi\right]\frac{du}{d\xi}-\left[\frac{|m|'+1}{2}-\frac{E}{2\hbar\omega_0}\right]u=0$$

$\rightarrow$ Confluent hypergeometric equation, $\Rightarrow \frac{|m'|+1}{2}-\frac{E}{2\hbar\omega_0}=-n_r$

$$E_{n_r,m}=(2n_r+|m'|+1)\hbar\omega_0$$

$$\psi_{n_r,m'}(r,\varphi)\sim e^{im'\varphi}\,r^{|m'|}\,e^{-\frac{r^2}{2l^2}}\,F\left(-n_r,|m'|+1,\frac{r^2}{l^2}\right)$$

$$F(\alpha, \gamma, z) = 1 + \frac{\alpha}{\gamma} z + \frac{\alpha(\alpha+1)}{2!\,\gamma(\gamma+1)} z^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{3!\,\gamma(\gamma+1)(\gamma+2)} z^3 + \cdots$$

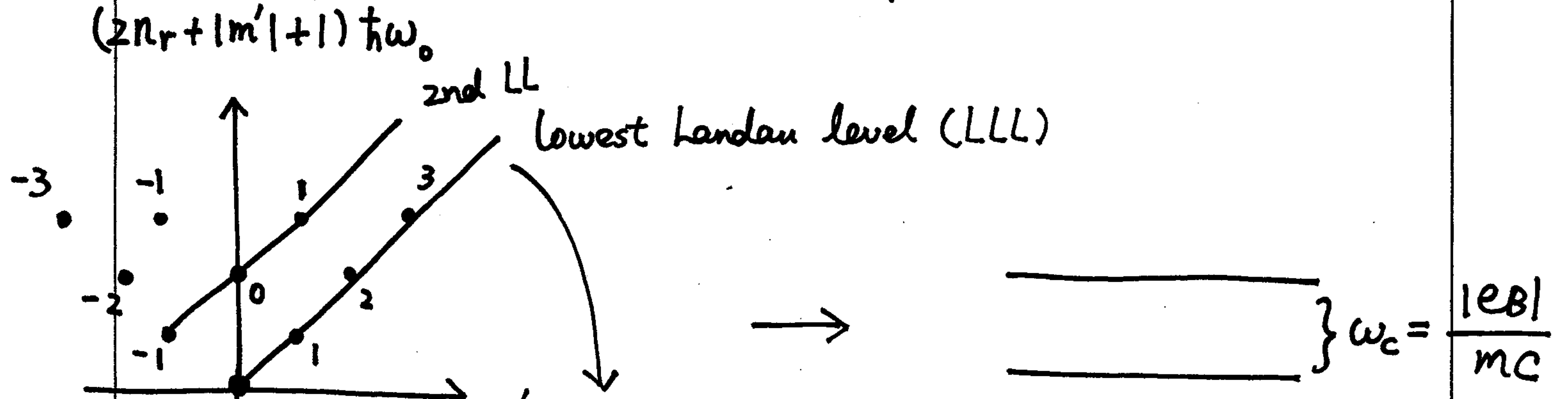
$$F\left(-n_r, |m'|+1, \frac{r^2}{\ell^2}\right) = 1 + \left(\frac{-n_r}{|m'|+1}\right)\frac{r^2}{\ell^2} + \frac{(-n_r)(-n_r+1)}{2!\,(|m'|+1)(|m'|+2)}\left(\frac{r}{\ell}\right)^4$$

$$+ \frac{(-n_r)(-n_r+1)\cdots(-1)}{n_r!\,(|m'|+1)(|m'|+2)\cdots(|m'|+n_r)}\left(\frac{r}{\ell}\right)^{2n_r} \leftarrow \text{Polynomial}$$

only has finite terms.

$$E_{Landau} = E_{n_r, m'} - m'\hbar\omega_0 = (2n_r + |m'| + 1 - m')\hbar\omega_0$$

$$= \begin{cases} \hbar\omega_c\,[n_r + 1/2] & \text{if } m' > 0 \\ \hbar\omega_c\,[n_r + |m'| + 1/2] & \text{if } m' < 0 \end{cases} \qquad \omega_c = 2\omega_0 = \frac{|eB|}{mc}$$



$$\ell_B = \sqrt{\frac{\hbar c}{eB}} = \frac{1}{\sqrt{2}}\ell$$

what's special of the lowest Landau level?

$$\psi^{LLL}_{n_r=0, m'} = \frac{1}{\sqrt{2\pi \ell_B^2\, 2^{m'} m'!}} \left(\frac{z}{\ell_B}\right)^{m'} e^{-\frac{|z|^2}{4\ell_B^2}}, \text{ where } z = x+iy$$

infinite degeneracy with respect to $m' = 0, 1, 2, 3, \cdots\cdots$

Classic radius (LLL states)

$$\rho = |\psi|^2 \propto r^{2m} e^{-\frac{r^2}{2\ell_B^2}} \Rightarrow \frac{\partial \rho}{\partial (r^2)} = \left[m (r^2)^{m-1} + (r^2)^m \left(-\frac{1}{2\ell_B^2}\right) \right] e^{-\frac{r^2}{2\ell_B^2}} = 0$$

$$\Rightarrow r_c^2 = 2m\,\ell_B^2$$

between the m-th, and the m+1-th orbits, there's one state

$$\Rightarrow \underset{\text{average density}}{\rho} \sim \frac{1}{\pi (r_c^2(m+1) - r_c^2(m))} = \frac{1}{2\pi \ell_B^2}$$

[Consider we fill the system with fermions)

Actually, this is exact result if LLL is fully filled!

$$\rho(r) = \sum_{m=0}^{\infty} |\psi_{LLL,m}(r)|^2 = \frac{1}{2\pi \ell_B^2} \left[\underbrace{\sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{|z|^2}{2\ell_B^2}\right)^m}_{e^{\frac{|z|^2}{2\ell_B^2}}} \right] e^{-\frac{|z|^2}{2\ell_B^2}}$$

$$= \frac{1}{2\pi \ell_B^2},$$ which is a const

Magnetic translation

the mechanical momenta are $\vec{\mathbb{P}} = \vec{p} - \frac{e}{c}\vec{A}$

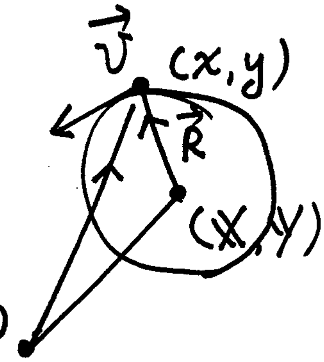
$$\Rightarrow \mathbb{P}_x = -i\hbar\partial_x + \frac{eB}{2c} y = -i\hbar\partial_x + \frac{\hbar y}{2\ell_B^2}$$

$$\mathbb{P}_y = -i\hbar\partial_y - \frac{eB}{2c} x = -i\hbar\partial_y - \frac{\hbar x}{2\ell_B^2}$$

define guiding center

$$\mathbb{X} = x + \frac{l_B^2}{\hbar} \mathbb{P}_y = \frac{l_B^2}{\hbar}\left(-i\hbar \partial_y + \frac{\hbar}{2l_B^2} x\right)$$

$$\mathbb{Y} = y - \frac{c}{eB} \mathbb{P}_x = -\frac{l_B^2}{\hbar}\left(-i\hbar \partial_x - \frac{\hbar}{2l_B^2} y\right)$$



$$H^{LL}_{2D} = \frac{\mathbb{P}_x^2}{2m} + \frac{\mathbb{P}_y^2}{2m}$$

check that the guiding centers are conserved quantities

$$[\mathbb{X}, H^{LL}] = [\mathbb{Y}, H^{LL}] = 0.$$

$$\vec{R} = \frac{c}{eB}(\mathbb{P}_y, -\mathbb{P}_x) = \frac{l_B^2}{\hbar}(\mathbb{P}_y, -\mathbb{P}_x)$$

Thus we can put the guiding center onto exponential

$$T_x(\delta_x) = e^{+i\frac{\mathbb{Y}\delta_x}{l_B^2}} = e^{-i(-i\partial_x - \frac{y}{2l_B^2})\cdot\delta_x} = e^{-\delta_x \partial_x + i\frac{y\delta_x}{2l_B^2}}$$

$$T_y(\delta_y) = e^{-\delta_y \partial_y - i\frac{x\delta_y}{2l_B^2}}$$

(δ_x, δ_y are translation distance)

$$[T_x(\delta_x), H^{LL}_{2D}] = [T_y(\delta_y), H^{LL}_{2D}] = 0$$

generally speaking, for a translation along a vector $\vec{\delta}$

$$\boxed{\begin{aligned} T[\vec{\delta}] &= e^{-\vec{\delta}\cdot\vec{\nabla} + \frac{i}{2l_B^2}\hat{z}\cdot(\vec{\delta}\times\vec{r})} \\ [T[\vec{\delta}], H^{LL}_{2D}] &= 0 \end{aligned}}$$

But $T_x[\delta_x]$ and $T_y[\delta_y]$ donot commute

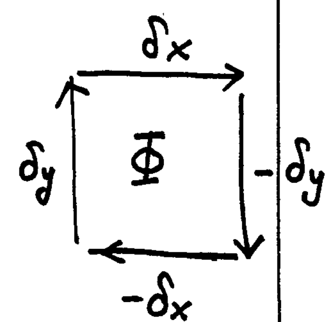
$$T_x[\delta_x]\, T_y[\delta_y] = e^{\delta_x[-\partial_x + \frac{iy}{2\ell_B^2}]}\, e^{\delta_y[-\partial_y - \frac{ix}{2\ell_B^2}]}$$

according to the fomula $e^A e^B = e^B e^A e^{[A,B]}$

if $[[A,B],A] = [[A,B],B] = 0$. We can check that $\boxed{[-\partial_x + \frac{iy}{2\ell_B^2}, -\partial_y - \frac{ix}{2\ell_B^2}] = \frac{i}{\ell_B^2}}$

$$\Rightarrow \quad T_x[\delta_x]\, T_y[\delta_y] = T_y[\delta_y]\, T_x[\delta_x]\, e^{i\frac{\delta_x \delta_y}{\ell_B^2}}$$

$$\Rightarrow \quad T_x^{-1}[\delta_x]\, T_y^{-1}[\delta_y]\, T_x[\delta_x]\, T_y[\delta_y] = e^{i\,\delta_x\delta_y/\ell_B^2}$$



$$= e^{i2\pi\, \Phi/\Phi_0}$$

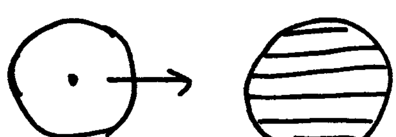
Let us try to ~~translate~~ the Gaussian pocket $\psi_{LLL,m=0} \propto e^{\frac{-|z|^2}{4\ell_B^2}}$

$$T_x[\delta]\, e^{\frac{-|z|^2}{4\ell_B^2}} = e^{-\delta\partial_x + \frac{i}{2\ell_B^2}\delta y}\, e^{-\frac{|z|^2}{4\ell_B^2}}$$

$$= e^{-\frac{(x-\delta)^2 + y^2}{4\ell_B^2}}\, e^{i\frac{y\delta}{2\ell_B^2}} = e^{\frac{-|z|^2}{4\ell_B^2}} \cdot \underbrace{e^{\frac{\delta}{2\ell_B^2}(x+iy)}}$$

$f(z)$: only depenents on z.

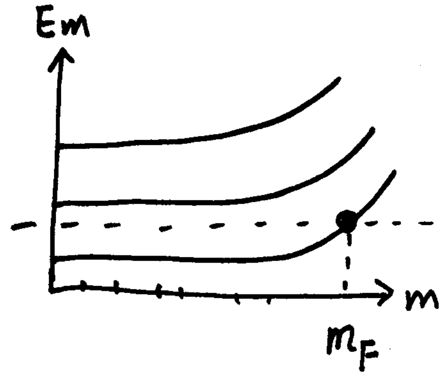
after translation, the state remains in the LLL.



$e^{i\, y\delta/2\ell_B^2}$

edge - spectra

Now let us consider put the system in a disc with radius R. What will be the spectra in this case with the open boundary condition?



as m is small, $r_{c,m} = \sqrt{2m}\, l_B << R$, thus for these states, they don't see the open boundary, thus their spectra remain flat.

However. for those states with large values of m, its classic radius $\rightarrow$ the wall, then their energy ~~s~~ are pushed up. As m goes very large, their wavefunctions are thrown to the wall, thus they are called edge states. The crietion is $\sqrt{2m}\, l_B >> R$. for these m's, they are edge states. In this case, its hamiltonian becomes a circular rotor,

$$H_{2D} = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 r^2 - \omega_0 L_z \qquad \boxed{m\hbar = Rk}$$

liner momentum

$$\underset{\text{edge}}{H_{1D}} \rightarrow \frac{m^2\hbar^2}{2MR^2} + \text{const} - m\hbar\omega_0 \qquad (m > 0)$$

we will fill with fermions, at chemical potential μ. If it cuts the spectra at $m \simeq m_F$, we can linearize the spectra as

$$H_{1D} \simeq \frac{v}{R}(m - m_F)\hbar \longrightarrow v(k - k_f) \leftarrow \text{Chiral Fermi liquid.}$$

if we consider edge as flat