Lecture 13 Addition of angular momentum

1. $\vec{J} = \vec{J}_1 + \vec{J}_2$ remains angular momentum

2. $j = j_1+j_2,\ j_1+j_2-1,\ \cdots\ |j_1-j_2|$

3. $|j_1 j_2 ; j m\rangle = \sum\limits_{m_1 m_2} |j_1 m_1\rangle |j_2 m_2\rangle \langle j_1 m_1\, j_2 m_2 | j m\rangle$

$$|j_1 m_1\rangle |j_2 m_2\rangle = \sum_{jm} |j_1 j_2 ; j m\rangle \langle j m | j_1 m_1\, j_2 m_2\rangle$$

§ Angular momentum addition

$$[J_{1i}, J_{1j}] = i\epsilon_{ijk} J_{1k}, \qquad [J_{2i}, J_{2j}] = i\epsilon_{ijk} J_{2k}$$

then $\vec{J} = \vec{J}_1 + \vec{J}_2$ remains an angular momentum

$[J_i, J_j] = i\epsilon_{ijk} J_k$. Nevertheless $\vec{J}_1 - \vec{J}_2$ is no longer an angular momentum.

Consider two sets of eigenstates $|j_1 m_1\rangle$ for J_1^2, J_{1z}, and $|j_2 m_2\rangle$ for J_2^2, J_{2z}. The direct product $|j_1 m_1 j_2 m_2\rangle$

$$J_{1,2}^2 |j_1 m_1\rangle|j_2 m_2\rangle = j_{1,2}(j_{1,2}+1)\,|j_1 m_1\rangle|j_2 m_2\rangle,$$

$$J_{1,2,z} |j_1 m_1\rangle|j_2 m_2\rangle = m_{1,2}\,|j_1 m_1\rangle|j_2 m_2\rangle.$$

Then we would like to superpose them to form the eigenstates $|jm\rangle$ satisfying

$$J^2 |jm\rangle = j(j+1)|jm\rangle$$

$$J_z |jm\rangle = m|jm\rangle.$$

We formally write down

$$|j_1 j_2; jm\rangle = \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | jm\rangle |j_1 m_1\rangle |j_2 m_2\rangle$$

where $\langle j_1 m_1 j_2 m_2 | jm\rangle$ is call the C-G coefficient (Clebsch-Gordon)

Reversely, $|j_1 m_1\rangle|j_2 m_2\rangle = \sum_{jm} |j_1 j_2; jm\rangle\langle jm|j_1 m_1 j_2 m_2\rangle$

$$\sum_{m_1 m_2} \langle jm | j_1 m_1 j_2 m_2 \rangle \langle j_1 m_1 j_2 m_2 | j' m' \rangle = \delta_{jj'} \delta_{mm'}$$

$$\sum_{jm} \langle j_1 m_1 j_2 m_2 | jm \rangle \langle jm | j_1 m_1' j_2 m_2' \rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

§: The values of j.

we have

$J_z |j_1 m_1 j_2 m_2\rangle = (m_1 + m_2) |j_1 m_1 j_2 m_2\rangle$. $(2j_1+1)(2j_2+1)$ bases.

① hence, the maxial vaule of $J_z = j_1 + j_2$, then the associated $J_{max} = j_1 + j_2$. This sector generates a set of $2(j_1+j_2)+1$ fold multiplets.

② Then for $J_z = j_1 + j_2 + 1$, there are 2 possible states.

$|j_1 j_1 - 1\rangle \otimes |j_2 j_2\rangle$, $|j_1 j_1\rangle \otimes |j_2 j_2 - 1\rangle$

The sector of $J = j_1 + j_2$ generates one $|j_1+j_2, j_1+j_2-1\rangle$ then there should exist another new state, whose J must be $j_1 + j_2 - 1$.

③ Then for the sector of $J_z = j_1 + j_2 - 2$, there exist 3 states $|j_1 j_1 - 2\rangle |j_2 j_2\rangle$ $|j_1 j_1 - 1\rangle |j_2 j_2 - 1\rangle$ $|j_1 j_1\rangle |j_2 j_2 - 2\rangle$

we have $J = j_1 + j_2$, $j_1 + j_2 - 1$, they yield 2 states with $J_z = j_1 + j_2 - 2$. Hence, the one left must belong to the total $J = j_1 + j_2 - 2$.

We can repeat this procedure, until $J = |j_1 - j_2|$.

We count the # of states: (assume $j_1 > j_2$)

$$J = j_1 + j_2 \quad \rightarrow \quad 2(j_1+j_2)+1$$

$$J = j_1 + j_2 - 1 \qquad 2(j_1+j_2-1)+1$$

$$\vdots$$

$$J = j_1 - j_2 \qquad 2(j_1-j_2)+1$$

$$\Sigma = (2j_1+2j_2+1 + 2j_1-2j_2+1) \times (j_1+j_2-j_1+j_2+1)/2 = (2j_2+1)(2j_1+1)$$

$J = j_1+j_2 \qquad J = j_1+j_2,\ j_1+j_2-1$

$$\cancel{|j_1 j_1\rangle |j_2 j_2\rangle}, \quad \cancel{|j_1 j_1-1\rangle |j_2 j_2\rangle}, \quad \cdots\cdots \quad |j_1 -j_1\rangle |j_2 j_2\rangle$$

$$\cancel{|j_1 j_1\rangle |j_2 j_2-1\rangle} \quad |j_1 j_1-1\rangle |j_2 j_2-1\rangle \qquad |j_1 -j_1\rangle |j_2 j_2-1\rangle$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$|j_1 j_1\rangle |j_2 -j_2\rangle \quad |j_1 j_1-1\rangle |j_2 -j_2\rangle \quad \cdots \quad |j_1 -j_1\rangle |j_2 -j_2\rangle$$

§ Example of spin ½, $\frac{1}{2} \times \frac{1}{2} = 0 + 1$.

spin ½ $\quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$$\vec{S} = \frac{\vec{\sigma}}{2}. \qquad S_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Consider two electrons each with spin ½, what's their total spin?

$$\vec{S} = \vec{S}_1 + \vec{S}_2 \quad \Rightarrow \quad S_- = S_{1-} + S_{2-}$$

① $|\tfrac{1}{2}\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 = |11\rangle$

$$(S_{1-} + S_{2-})\, |\tfrac{1}{2}\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 = S_- |11\rangle = \sqrt{(1+1)(1-1+1)}\, |10\rangle$$

② $|\tfrac{1}{2}-\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 + |\tfrac{1}{2}\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}-\tfrac{1}{2}\rangle_2 = \sqrt{2}\, |10\rangle$

$$(S_{1-} + S_{2-})\, \left(|\tfrac{1}{2}-\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 + |\tfrac{1}{2}\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}-\tfrac{1}{2}\rangle_2 \right) = \sqrt{2}\, S_- |10\rangle$$

$$|\tfrac{1}{2}-\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}-\tfrac{1}{2}\rangle_2 + |\tfrac{1}{2}-\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}-\tfrac{1}{2}\rangle_2 = \sqrt{2}\, \sqrt{(1-0)(1-0+1)}\, |1-1\rangle$$

$$\Rightarrow \quad |\tfrac{1}{2}-\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}-\tfrac{1}{2}\rangle_2 = |1-1\rangle$$

← in simplified notation

Summary:
$$\begin{cases} |11\rangle = |\uparrow\rangle_1 |\uparrow\rangle_2 \\ |10\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2 \right) \\ |1-1\rangle = |\downarrow\rangle_1 |\downarrow\rangle_2 \end{cases}$$

Then we have another state with $S_z = 0$, which is orthogonal to $|10\rangle$. This state should belong to $|00\rangle \propto |\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2$

But we need to assign a sign convention, we require that

$$\langle j' j_z^{(12)} | j_{1z} | j j_z^{(12)} \rangle \geq 0.$$

Here, $|j j_z^{(12)}\rangle$ means the construction by a fixed sequence of J_1 and J_2. If we switch the sequence to be J_2 and J_1, then the consequence is denoted as $|j j_z^{(21)}\rangle$.

According to this convention, we have

$$\boxed{|00\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right)}$$

such that $\langle 00|j_{1z}|10\rangle = \frac{1}{2}\left(\langle\uparrow|_1\langle\downarrow|_2 - \langle\downarrow|_1\langle\uparrow|_2\right) \cdot \left(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right) > 0$

Then we have obtained all the $\langle 1m|\tfrac{1}{2}s_z\tfrac{1}{2}s_z'\rangle$.
CG coefficents of

§ how about $1\times\tfrac{1}{2} = \tfrac{3}{2}\oplus\tfrac{1}{2}$

$$|\tfrac{3}{2}\tfrac{3}{2}\rangle = |11\rangle_1\otimes|\tfrac{1}{2}\tfrac{1}{2}\rangle_2 \quad \downarrow J_-$$

$$|\tfrac{3}{2}\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|10\rangle_1|\tfrac{1}{2}\tfrac{1}{2}\rangle_2 + \sqrt{\tfrac{1}{3}}\,|11\rangle_1|\tfrac{1}{2}-\tfrac{1}{2}\rangle_2 \quad \downarrow J_-$$

$$|\tfrac{3}{2}-\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|1-1\rangle_1|\tfrac{1}{2}\tfrac{1}{2}\rangle_2 + \sqrt{\tfrac{2}{3}}\,|10\rangle_1|\tfrac{1}{2}-\tfrac{1}{2}\rangle_2 \quad \downarrow J_-$$

$$|\tfrac{3}{2}-\tfrac{3}{2}\rangle = |1-1\rangle_1\otimes|\tfrac{1}{2}-\tfrac{1}{2}\rangle_2$$

another state with $J_z = \tfrac{1}{2}$, $\sqrt{\tfrac{1}{3}}\,|10\rangle_1|\tfrac{1}{2}\tfrac{1}{2}\rangle_2 - \sqrt{\tfrac{2}{3}}\,|11\rangle_1|\tfrac{1}{2}-\tfrac{1}{2}\rangle$

$$j_{1z}|\tfrac{3}{2}\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|11\rangle_1|\tfrac{1}{2}-\tfrac{1}{2}\rangle_2$$

$$\langle\tfrac{1}{2}\tfrac{1}{2}|j_z|\tfrac{3}{2}\tfrac{1}{2}\rangle > 0 \Rightarrow |\tfrac{1}{2}\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|11\rangle_1|\tfrac{1}{2}-\tfrac{1}{2}\rangle_2 - \sqrt{\tfrac{1}{3}}\,|10\rangle_1|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$|\tfrac{1}{2}-\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|10\rangle_1|\tfrac{1}{2}-\tfrac{1}{2}\rangle_2 - \sqrt{\tfrac{2}{3}}\,|1-1\rangle_1|\tfrac{1}{2}\tfrac{1}{2}\rangle$$