Runge - Lentz vector (elementry knowledge).

① Classic kepler problem

$$H = \frac{p^2}{2m} - \frac{k}{r}$$

obviouly $\dot{\vec{L}} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = 0$ for centeral force fields.

how about $\frac{d}{dt}(\vec{p} \times \vec{L}) = \dot{\vec{p}} \times \vec{L} + \vec{p} \times \dot{\vec{L}}$

$$\dot{\vec{p}} \times \vec{L} = \frac{-k}{r^2} \hat{r} \times (\vec{r} \times m\dot{\vec{r}}) = -\frac{k}{r^2} m [\vec{r}(\hat{r} \cdot \dot{\vec{r}}) - \hat{r} \cdot \vec{r}\, \dot{\vec{r}}]$$

$$= -\frac{k}{r^3} m [\vec{r}[\vec{r} \cdot \dot{\vec{r}}] - r^2 \dot{\vec{r}}] \leftarrow \vec{r}\dot{\vec{r}} = r\dot{r}$$

$$= -k\, m \left[\frac{\vec{r}\,\dot{r}}{r^2} - \frac{\dot{\vec{r}}}{r}\right]$$

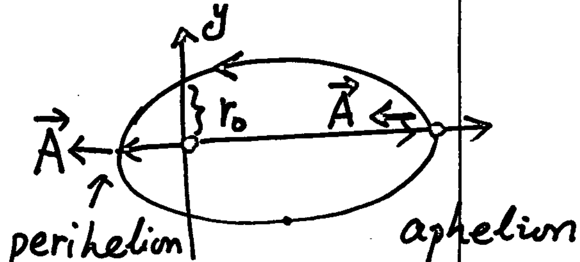
$$\Rightarrow \frac{d}{dt}(\vec{p} \times \vec{L}) = +km\left[-\frac{\vec{r}\,\dot{r}}{r^2} + \frac{\dot{\vec{r}}}{r}\right] = km \frac{d}{dt}\left[\frac{\vec{r}}{r}\right]$$

define $\vec{A} = \vec{p} \times \vec{L} - mk\frac{\vec{r}}{r} \Rightarrow \frac{d}{dt}\vec{A} = 0$. $\vec{A}$: Runge-lentz vector.

what's the direction of $\vec{A}$?

$$\vec{A} \cdot \vec{L} = (\vec{p} \times \vec{L}) \cdot \vec{L} - mk\frac{\vec{r}}{r} \cdot (\vec{r} \times \vec{p}) = 0,$$

thus $\vec{A}$ lies in the orbital plane.



check at perihelim:

$$\vec{p} = -p\hat{y}, \quad \vec{L} = l\hat{z}$$

$$\vec{p} \times \vec{L} = -pl\,\hat{x}$$

and $r = \frac{r_0}{1 - e\cos\theta}$

Kepler's problem:
$$\begin{cases} r_0 = \frac{l^2}{mk} \\ e = \sqrt{1 + \frac{2El^2}{mk^2}} = \sqrt{1 + \frac{2E}{k/r_0}} \end{cases}$$

$$P\cdot\frac{r_0}{1+e} = \ell \Rightarrow P\cdot\ell = \frac{\ell^2(1+e)}{r_0} = mk(1+e)$$

$$\Rightarrow \vec{A} = -P\ell\,\hat{x} + \underset{mk}{\hat{x}} = \left[-mk(1+e) + mk\right]\hat{x} = -mke\,\hat{x}$$

A's magnitude is $\propto$ eccentricity; it's direction points along to the perihelion.

Let's check $\vec{A}\cdot\vec{r} = -Ar\cos\theta = \vec{r}\cdot(\vec{p}\times\vec{L}) - mkr$

$$\vec{r}\cdot(\vec{p}\times\vec{L}) = \vec{L}\cdot(\vec{r}\times\vec{p}) = \ell^2$$

$$\Rightarrow -Ar\cos\theta = \ell^2 - mkr \Rightarrow \boxed{r = \frac{\ell^2}{mk - A\cos\theta} = \frac{\ell^2/mk}{1-\frac{A}{mk}\cos\theta}}$$

Solution to Kepler problem using R-L vector

compare $e = \sqrt{1+\frac{2E\ell^2}{mk^2}} = \frac{A}{mk} \Rightarrow \boxed{\frac{A^2}{m^2k^2} = 1 + \frac{2E\ell^2}{m^2k^2}}$

once E, $\vec{L}$ are determined,

$|A|$, and $\vec{A}\perp\vec{L}$ are decided, there's only one more angle to determine the direction of $\vec{A}$. Once this is determined, the orbit is determined.

Count degree of freedom: $E, \vec{L}$: 4

direction of $\vec{A}$: $1 \Leftarrow \left\{ \begin{array}{l} \vec{A}\cdot\vec{L} = 0 \\ A^2 = m^2k^2 + 2E\ell^2 \end{array} \right.$

initial condition: 1

6 (complete!)

2. 2D Quantum mechanics Kepler problem

$$\vec{A} = \frac{1}{2mk}(\vec{p}\times\vec{l} - \vec{l}\times\vec{p}) - \hat{e}_r \quad \text{(a different normalization)}$$

in 2D: $A_x = \frac{1}{2mk}(P_y l_z + l_z P_y) - \hat{e}_{r,x}$, $A_y = \frac{1}{2mk}(P_x l_z + l_z P_x) - \hat{e}_{r,y}$

$$\Rightarrow \boxed{A_x = \frac{1}{mk} l_z P_y + \frac{i\hbar}{2mk} P_x - \frac{x}{r}\ ; \quad A_y = -\frac{1}{mk} l_z P_x + \frac{i\hbar}{2mk} P_y - \frac{y}{r}}$$

$A_{x,y}$ are Hermitian, but $l_z P_y$ and $l_z P_x$ are not. Thus $\frac{i\hbar}{2mk}\vec{P}$ comes to compensate

⊛ $\boxed{\text{Check } [l_z A_x] = i\hbar A_y \text{ and } [l_z, A_y] = -i\hbar A_x}$

$$[l_z, A_x] = \frac{1}{mk}[l_z, l_z P_y] + \frac{i\hbar}{2mk}[l_z, P_x] - [l_z, \frac{x}{r}]$$

$$= \frac{1}{mk} l_z(-i\hbar P_x) + \frac{i\hbar}{2mk}(i\hbar P_y) - i\hbar\frac{y}{r} = i\hbar A_y$$

the other one can be proved similarly.

⊛ $\boxed{\text{Check } [A_x, A_y] = -\frac{2H}{mk^2} i\hbar l_z}$

$$[A_x, A_y] = -\frac{1}{(mk)^2}[l_z P_y, l_z P_x] + \frac{i\hbar}{2(mk)^2}\{[l_z P_y, P_y] + [l_z P_x, P_x]\}$$

$$-\frac{i\hbar}{2mk}\{[P_x, \frac{y}{r}] - [P_y, \frac{x}{r}]\} - \frac{1}{mk}\{[L_z P_y, \frac{y}{r}] + [l_z P_x, \frac{x}{r}]\}$$

$$[l_z P_y, l_z P_x] = l_z[P_y, l_z P_x] + [l_z, l_z P_x]P_y = l_z[P_y\, l_z]P_x + l_z[l_z, P_x]P_y$$

$$= l_z(i\hbar)P_x^2 + i\hbar l_z P_y^2 = i\hbar l_z P^2$$

$$[l_z P_y, P_y] = P_y[l_z, P_y] = -i\hbar P_y P_x, \quad [l_z P_x, P_x] = P_x[l_z, P_x] = i\hbar P_x P_y \xrightarrow{\text{add together.}} 0$$

$$[P_x, \frac{y}{r}] = -i\hbar\, y\,\partial_x \frac{1}{r} = i\hbar\frac{xy}{r^3}, \quad [P_y, \frac{x}{r}] = -i\hbar\, x\,\partial_y\frac{1}{r} = -i\hbar\frac{xy}{r^3} \xrightarrow{\text{add together}} 0$$

$$[l_z P_y, \frac{y}{r}] = l_z [P_y \frac{y}{r}] + [l_z, \frac{y}{r}] P_y = l_z (-i\hbar) \partial_y \frac{y}{r} - i\hbar \frac{x}{r} P_y$$

$$\partial_y \frac{y}{r} = \frac{1}{r} - \frac{y^2}{r^3} \quad \rightarrow \quad = (-i\hbar) l_z (\frac{1}{r} - \frac{y^2}{r^3}) - i\hbar \frac{1}{r} x P_y$$

$$[l_z P_x, \frac{x}{r}] = -i\hbar l_z (\frac{1}{r} - \frac{x^2}{r^3}) - i\hbar \frac{y}{r} P_x$$

add together $\Rightarrow -i\hbar l_z (\frac{2}{r} - \frac{1}{r}) - i\hbar \frac{1}{r} l_z = -2i\hbar l_z \frac{1}{r}$

organize everything $\Rightarrow [A_x, A_y] = -\frac{1}{(mk)^2} i\hbar l_z p^2 + \frac{1}{mk} i\hbar l_z \frac{2}{r}$

$$= \frac{-2}{mk^2} [\frac{p^2}{2m} - \frac{k}{r}] l_z i\hbar = -\frac{2H}{mk^2} i\hbar l_z$$

⊛ Next prove $\boxed{[A_x, H] = [A_y, H] = 0.}$

$$[A_x, H] = \frac{1}{2m} [A_x, p^2] - k [A_x, \frac{1}{r}]$$

we need $[\vec{p} \frac{1}{r}] = -i\hbar \nabla \frac{1}{r} = +i\hbar \frac{\vec{r}}{r^3}$,

$$\nabla \cdot (\frac{\vec{r}}{r^3}) = \frac{1}{r^3} \nabla \cdot \vec{r} + \nabla (\frac{1}{r^3}) \vec{r} = \frac{2}{r^3} + \frac{-3\vec{r}\cdot\vec{r}}{r^5} = -\frac{1}{r^3}$$

$$[A_x, p^2] = \frac{1}{mk} [\underset{\uparrow \atop 0}{l_z P_y, p^2}] - [\frac{x}{r}, p^2] = [p^2, \frac{x}{r}]$$

$$[p^2, \frac{x}{r}] = x [p^2 \frac{1}{r}] + [p^2, x] \frac{1}{r} = x \vec{p} [\vec{p} \frac{1}{r}] + x [\vec{p}, \frac{1}{r}] \vec{p} + \vec{p} [\vec{p} x] \frac{1}{r} + [\vec{p}, x] \vec{p} \frac{1}{r}$$

$$= x i\hbar [\vec{p} \cdot \vec{r} / r^3 + \frac{\vec{r}}{r^3} \cdot \vec{p}] - 2i\hbar P_x \frac{1}{r}$$

$$= 2i\hbar x \frac{\vec{r}}{r^3} \cdot \vec{p} + x i\hbar (-i\hbar) (\frac{-1}{r^3}) - \underbrace{2i\hbar (-i\hbar) \frac{-x}{r^3} - 2i\hbar \frac{1}{r} P_x}$$

$$= \boxed{2i\hbar x [-\frac{1}{r} P_x + \frac{x}{r^3} \vec{r} \cdot \vec{p}] + \frac{\hbar^2 x}{r^3} = [A_x, p^2]}$$

$$[A_x \tfrac{k}{r}] = \frac{1}{m}[l_z P_y \tfrac{1}{r}] + \frac{i\hbar}{2m}[P_x \tfrac{1}{r}]$$

$$= \frac{1}{m} l_z [P_y \tfrac{1}{r}] + \frac{i\hbar}{2m}[P_x \tfrac{1}{r}] = \frac{1}{m} l_z(-i\hbar)\partial_y \frac{1}{r} + \frac{i\hbar}{2m}(-i\hbar)\partial_x \frac{1}{r}$$

$$= \frac{1}{m} l_z(-i\hbar)\frac{-y}{r^3} - \frac{\hbar^2}{2m}\frac{x}{r^3} = \frac{+i\hbar}{m}\Big(\frac{y}{r^3} l_z + \underset{-i\hbar x}{\underset{\|}{[l_z, y]}}\Big) - \frac{\hbar^2}{2m}\frac{x}{r^3}$$

$$= \frac{i\hbar}{m}\frac{1}{r^3}[y(xP_y - yP_x)] + \frac{\hbar^2}{2m}\frac{x}{r^3}$$

$$= \frac{\hbar^2}{m}\frac{x}{r^3}(x\partial_x + y\partial_y) - \frac{\hbar^2}{m}\frac{1}{r^3}(x^2+y^2)\partial_x + \frac{\hbar^2}{2m}\frac{x}{r^3}$$

$$\boxed{[A_x, \tfrac{k}{r}] = \frac{i\hbar^2}{m}\left[\frac{x}{r^3}(\vec{r}\cdot\vec{p}) - \frac{1}{r}P_x\right] + \frac{\hbar^2}{2m}\frac{x}{r^3}}$$

add ~~together~~ $\Rightarrow$ $[A_x, H] = 0$, and similarly $[A_y, H] = 0$.

$\boxed{\left\{\sqrt{\frac{-mk^2}{2E}}A_x,\ \sqrt{\frac{-mk^2}{2E}}A_y,\ l_z\right\} \text{ form an SU(2) algebra.}}$

(*) Calculate Casimar

$$A_x^2 + A_y^2 = \Big(\frac{1}{mk}l_zP_y + \frac{i\hbar}{2mk}P_x - \frac{x}{r}\Big)^2 + \Big(\frac{-1}{mk}l_zP_x + \frac{i\hbar}{2mk}P_y - \frac{y}{r}\Big)^2$$

$$= \frac{1}{m^2k^2}l_z^2P^2 - \frac{\hbar^2}{4mk^2}P^2 + 1 + \frac{i\hbar}{2(mk)^2}[\cancel{l_zP_y}P_x + P_xl_zP_y - \cancel{l_zP_x}P_y - P_yl_zP_x]$$

$$-\frac{1}{mk}\Big(l_zP_y\frac{x}{r} - l_zP_x\frac{y}{r} + \frac{x}{r}l_zP_y - \frac{y}{r}l_zP_x\Big)$$

$$-\frac{i\hbar}{2mk}\Big(P_x\frac{x}{r} + P_y\frac{y}{r} + \frac{x}{r}P_x + \frac{y}{r}P_y\Big)$$

$$l_zP_yl_zP_y + l_zP_xl_zP_x = l_z^2(P_y^2+P_x^2) + \underbrace{l_z[P_y\,l_z]P_y + l_z[P_x\,l_z]P_x}_{\text{add to 0.}} = l_z^2P^2$$

$$P_x l_z P_y - P_y l_z P_x = l_z P_x P_y - l_z P_y P_x + [P_x\, L_z] P_y - [P_y\, L_z] P_x$$

$$= 0 - i\hbar P_y^2 - i\hbar P_x^2 = -i\hbar P^2$$

$\therefore$ P^2 related terms $\frac{P^2}{m}\left[\frac{1}{mk^2}\right]\left[l_z^2 - \frac{\hbar^2}{4} + \frac{\hbar^2}{2}\right] = \frac{P^2}{m}\frac{1}{mk^2}\left[l_z^2 + \frac{\hbar^2}{4}\right]$

$$l_z P_y \frac{x}{r} - l_z P_x \frac{y}{r} + \frac{x}{r} l_z P_y - \frac{y}{r} l_z P_x$$

$$= l_z [x P_y - y P_x] \frac{1}{r} + \frac{1}{r} l_z [x P_y - y P_x] + \frac{1}{r}[X\, L_z] P_y - \frac{1}{r}[y\, L_z] P_x$$

$$= \frac{2}{r} l_z^2 + \frac{1}{r}(-i\hbar) y P_y + \frac{1}{r}(-i\hbar) x P_x = \frac{2}{r} l_z^2 - i\hbar \frac{1}{r} \vec{r}\cdot\vec{p}$$

$$P_x\left(\frac{x}{r}\right) + P_y\left(\frac{y}{r}\right) + \frac{x}{r} P_x + \frac{y}{r} P_y = 2\frac{1}{r}\vec{r}\cdot\vec{p} - i\hbar\, \partial_x\left(\frac{x}{r}\right) - i\hbar\, \partial_y\left(\frac{y}{r}\right)$$

$$\partial_x\left(\frac{x}{r}\right) = \frac{1}{r} + \frac{x(-x)}{r^3} \Rightarrow -i\hbar\partial_x\left(\frac{x}{r}\right) - i\hbar\partial_y\left(\frac{y}{r}\right) = -i\hbar\left[\frac{2}{r} - \frac{1}{r}\right] = \frac{-i\hbar}{r}$$

put ~~together~~ $\frac{1}{r}$-related terms are

$$-\frac{1}{mk}\left[\frac{2}{r} l_z^2 - i\hbar \cancel{\frac{1}{r}\vec{r}\cdot\vec{p}}\right] - \frac{i\hbar}{2mk}\left[\cancel{2\frac{1}{r}\vec{r}\cdot\vec{p}} - \frac{i\hbar}{r}\right]$$

$$= \frac{2}{mk^2}\frac{(-k)}{r}\left[l_z^2 + \frac{\hbar^2}{4}\right]$$

$$\Rightarrow \boxed{A_x^2 + A_y^2 = \frac{2H}{mk^2}\left(l_z^2 + \frac{\hbar^2}{4}\right) + 1}$$

The normalized Casimar is : (replace H with eigenvalue E).

$$\left(\sqrt{\frac{-mk^2}{2E}}\right)^2 A_x^2 + \left(\sqrt{\frac{-mk^2}{2E}}\right)^2 A_y^2 + l_z^2 = \frac{-mk^2}{2E} - \frac{\hbar^2}{4}$$ (l_z^2 term cancels)

because L_z is orbital angler mementum $\Rightarrow$ L can only be integer

$$= L(L+1)\hbar^2 \quad (L = 0, 1, 2, \cdots)$$

SU(2) quntum number

$$\Rightarrow \frac{-mk^2}{2E} = \hbar^2\left(L+\frac{1}{2}\right)^2 \Rightarrow E = \frac{-mk^2}{2(L+\frac{1}{2})^2} \quad (L=0,1,2,\dots)$$

The corresponding $J_\pm$ operater

$$J_\pm = \sqrt{\frac{-mk^2}{2E}}\,(A_x \pm iA_y) = \hbar(L+\tfrac{1}{2})(A_x \pm iA_y)$$

$$J_\pm |E\ m\rangle = \hbar\sqrt{(L\mp m)(L\pm m+1)}\ |E\ m\pm 1\rangle$$

$$\Rightarrow \boxed{\begin{array}{l}(A_x \pm iA_y)\,|E(L), m\rangle = \dfrac{1}{L+\frac{1}{2}}\sqrt{(L\mp m)(L\pm m+1)}\ |E, m\pm 1\rangle. \\ l_z |E m\rangle = m\hbar\,|E, m\rangle\end{array}}$$

3. 3D case $H = \frac{p^2}{2m} - \frac{k}{r}$

① $\vec{A} = \frac{1}{2mk}(\vec{p}\times\vec{l} - \vec{l}\times\vec{p}) - \hat{e}_r$

$$A_x = \frac{1}{2mk}(P_y l_z - P_x l_y + l_z P_y - l_y P_x) - \hat{e}_{r,x} = \frac{1}{mk}P_y l_z - \frac{i\hbar}{mk}P_x - \frac{x}{r}$$

$$\Rightarrow \vec{A} = \frac{1}{mk}[\ \vec{p}\times\vec{l} - i\hbar\vec{p}) - \hat{e}_r$$

$$\vec{p}\times\vec{l} = \vec{p}\times(\vec{r}\times\vec{p}) = P_x\vec{r}P_x + P_y\vec{r}P_y + P_z\vec{r}P_z - (\vec{p}\cdot\vec{r})\vec{p}$$
$$= P_x(P_x\vec{r} + i\hbar\hat{x}) + \cdots - (\vec{p}\cdot\vec{r})\vec{p}$$
$$= p^2\vec{r} + i\hbar\vec{p} - (\vec{p}\cdot\vec{r})\vec{p}$$

$$\Rightarrow \boxed{\vec{A} = \frac{1}{mk}(p^2\vec{r} - (\vec{p}\cdot\vec{r})\vec{p}) - \vec{r}/r}$$

② Prove $\vec{A}\times\vec{A} = -\frac{2i\hbar}{mk^2}H\vec{l}$

$$m^2k^2\ \vec{A}\times\vec{A} = (p^2\vec{r} - (\vec{p}\cdot\vec{r})\vec{p} - mk\,\vec{r}/r)\times(p^2\vec{r} - (\vec{p}\cdot\vec{r})\vec{p} - mk\,\vec{r}/r)$$

The trick is to move the second expression: $p^2\vec{r} \to \vec{r}p^2$ and so on.

$$p^2\vec{r} = P_i\vec{r}P_i + P_i[P_i\vec{r}] = P_i\vec{r}P_i + \vec{p}(-i\hbar) = \vec{r}p^2 + [P_i\vec{r}]P_i - (i\hbar)\vec{p}$$
$$= \vec{r}p^2 - 2i\hbar\vec{p}$$

$$(\vec{p}\cdot\vec{r})\vec{p} = P_i r_i P_j = P_i P_j r_i + P_i[r_i P_j] = P_i P_j r_i + i\hbar P_j = \vec{p}(\vec{p}\cdot\vec{r}) + i\hbar\vec{p}$$

$$\Rightarrow \vec{A}\times\vec{A}\,m^2k^2 = (p^2\vec{r} - (\vec{p}\cdot\vec{r})\vec{p} - mk\,\vec{r}/r)\times(\vec{r}p^2 - 3i\hbar\vec{p} - \vec{p}(\vec{p}\cdot\vec{r}) - mk\,\vec{r}/r)$$

$$p^2\vec{r}\times\vec{r}p^2 = (\vec{p}\cdot\vec{r})\vec{p}\times\vec{p}(\vec{p}\cdot\vec{r}) = mk\frac{\vec{r}}{r}\times\frac{\vec{r}}{r} = 0$$

$$\Rightarrow m^2k^2\,\vec{A}\times\vec{A} = p^2(-3i\hbar)\,\vec{r}\times\vec{p} - p^2(\vec{r}\times\vec{p})(\vec{p}\cdot\vec{r})$$
$$-(\vec{p}\cdot\vec{r})(\vec{p}\times\vec{r})p^2 + (\vec{p}\cdot\vec{r})(\vec{p}\times\vec{r}/r)(mk) - mk\frac{\vec{r}}{r}\times(-3i\hbar\vec{p})$$
$$+ mk\left(\frac{\vec{r}}{r}\times\vec{p}\right)(\vec{p}\cdot\vec{r})$$
$$= -3i\hbar p^2\vec{l} - p^2(\vec{p}\cdot\vec{r})\vec{l} + (\vec{p}\cdot\vec{r})\vec{l}p^2 - mk(\vec{p}\cdot\vec{r})\frac{1}{r}\vec{l} + 3i\hbar mk\frac{1}{r}\vec{l}$$
$$+ mk\frac{1}{r}(\vec{p}\cdot\vec{r})\vec{l}$$
$$= \left(-3i\hbar p^2 + [\vec{p}\cdot\vec{r}, p^2] - mk[\vec{p}\cdot\vec{r}, \tfrac{1}{r}] + \frac{3i\hbar mk}{r}\right)\vec{l}$$

($\vec{l}$ commutes with all rotation invariant operators)

$$[\vec{p}\cdot\vec{r}, p^2] = [P_i r_i\ P_jP_j] = P_i[r_i\ P_jP_j] = P_iP_j[r_i\ P_j] + P_i[r_i, P_j]P_j$$
$$= +i\hbar\, 2p^2$$
$$[\vec{p}\cdot\vec{r}\ \tfrac{1}{r}] = [P_i r_i\ \tfrac{1}{r}] = [P_i\ \tfrac{1}{r}]r_i = -i\hbar\frac{-\vec{r}\cdot\vec{r}}{r^3} = i\hbar\frac{1}{r}$$

$$\Rightarrow m^2k^2\,\vec{A}\times\vec{A} = \left(-3i\hbar p^2 + 2i\hbar p^2 - \frac{i\hbar mk}{r} + \frac{3i\hbar mk}{r}\right)\vec{l}$$
$$= -i\hbar\left[\frac{p^2}{2m} - \frac{k}{r}\right]\cdot 2m\vec{l} = -2i\hbar\, mH\vec{l}$$

$$\Rightarrow \boxed{\vec{A}\times\vec{A} = -\frac{2i\hbar}{mk^2}H\vec{l}}$$

③ Prove $\boxed{[L_i\ A_j] = i\epsilon_{ijk}A_k\hbar}$

This is obvious because A is defined in a 3-vector form, under spatial rotation A transforms like a vector.

This means $\boxed{L_{12} = L_z,\ L_{23} = L_x,\ L_{31} = L_y \text{ and } L_{14} = \sqrt{\frac{-mk^2}{2E}}A_x,\ L_{24} = \sqrt{\frac{-mk^2}{2E}}A_y,\ L_{34} = \sqrt{\frac{-mk^2}{2E}}A_z}$ form SO(4) algebra.

~~4 prove $[\vec{A}, H] = 0$~~

$$\vec{A} = \frac{1}{mk}(\vec{p}\times\vec{\ell} - i\hbar\vec{p}) - \vec{r}/r$$

$$[\vec{A}, p^2] = \frac{1}{2mk}[\vec{p}\times\vec{\ell} - \vec{\ell}\times\vec{p}, p^2] - [\frac{\vec{r}}{r}, p^2]$$

This first term is 0 becase $[\vec{\ell}, p^2] = 0$.

$$[p^2, \frac{\vec{r}}{r}] = \vec{r}[p^2, \frac{1}{r}] + [p^2, \vec{r}]\frac{1}{r}$$

$$[p^2, \frac{1}{r}] = 2i\hbar\frac{\vec{r}\cdot\vec{p}}{r^3} + \hbar^2\nabla\cdot\left(\frac{\vec{r}}{r^3}\right) = 2i\hbar\frac{\vec{r}\cdot\vec{p}}{r^3} + 4\pi\hbar^2\delta^{(3)}(\vec{r})$$

$$\Rightarrow \vec{r}[p^2, \frac{1}{r}] = 2i\hbar\frac{\vec{r}}{r^3}\vec{r}\cdot\vec{p} + 4\pi\hbar^2\vec{r}\,\underbrace{\delta^{(3)}(\vec{r})}_{\to 0}$$

$$[p^2, \vec{r}] = -2i\hbar\vec{p}, \quad [p^2, \vec{r}]\frac{1}{r} = -\frac{2i\hbar}{r}\vec{p} + 2\hbar^2\frac{\vec{r}}{r^3}$$

$$\Rightarrow [p^2, \frac{\vec{r}}{r}] = \boxed{2i\hbar\left[-\frac{1}{r}\vec{p} + \frac{\vec{r}}{r^3}\vec{r}\cdot\vec{p}\right] + 2\hbar^2\frac{\vec{r}}{r^3} = [\vec{A}, p^2]}$$

**

$$[\vec{A}, \frac{1}{r}] = \frac{1}{mk}[\vec{p}\times\vec{\ell} - i\hbar\vec{p}, \frac{1}{r}]$$

$$[\vec{p}\times\vec{\ell}, \frac{1}{r}] = \vec{p}\times[\vec{\ell}, \frac{1}{r}] + [\vec{p}, \frac{1}{r}]\times\vec{\ell} = [\vec{p}, \frac{1}{r}]\times\vec{\ell}$$

$$[\vec{p}, \frac{1}{r}] = i\hbar\frac{\vec{r}}{r^3}$$

$$\Rightarrow [\vec{A}, \frac{1}{r}] = \frac{1}{mk}i\hbar\frac{1}{r^3}\vec{r}\times(\vec{r}\times\vec{p}) - \frac{i\hbar}{mk}(i\hbar)\frac{\vec{r}}{r^3}$$

$$\vec{r}\times(\vec{r}\times\vec{p}) = r_x\vec{r}p_x + r_y\vec{r}p_y + r_z\vec{r}p_z - (\vec{r}\cdot\vec{r})\vec{p}$$
$$= \vec{r}(\vec{r}\cdot\vec{p}) - r^2\vec{p}$$

$$\Rightarrow \boxed{[\vec{A}\ \frac{k}{r}] = \frac{i\hbar}{m}\left[\frac{\vec{r}}{r^3}(\vec{r}\cdot\vec{p}) - \frac{\vec{p}}{r}\right] + \frac{\hbar^2}{m}\frac{\vec{r}}{r^3}}$$

$$\Rightarrow [\vec{A}, \frac{p^2}{2m} - \frac{k}{r}] = 0\ !$$

⑤ $\boxed{\text{Prove } A^2 = \frac{2H}{mk^2}(\vec{l}^2 + \hbar^2) + 1}$

$$m^2k^2 A^2 = (\vec{p}\times\vec{l} - i\hbar\vec{p} - mk\vec{r}/r)\cdot(\vec{p}\times\vec{l} - i\hbar\vec{p} - mk\vec{r}/r)$$

$$= (\vec{p}\times\vec{l})\cdot(\vec{p}\times\vec{l}) - \hbar^2 p^2 + (mk)^2$$

$$- i\hbar\left[(\vec{p}\times\vec{l})\cdot\vec{p} + \vec{p}\cdot(\vec{p}\times\vec{l})\right] - mk\left[(\vec{p}\times\vec{l})\cdot\vec{r}/r + \vec{r}/r\cdot(\vec{p}\times\vec{l})\right]$$

$$+ im\hbar k\left[\vec{p}\cdot\vec{r}/r + \vec{r}/r\cdot\vec{p}\right]$$

$$(\vec{p}\times\vec{l})\cdot(\vec{p}\times\vec{l}) = p_i l_j p_i l_j - p_i l_j p_j l_i = p_i p_i l_j l_j + p_i[l_j, p_i] l_j - p_i l_i l_j p_j$$

($l_j p_j$ is rotation scalar)

$$= p^2 l^2 - \epsilon_{ijk} i\hbar \; p_i p_k l_j - p_i l_i p_j l_j = \boxed{p^2 l^2 - (\vec{p}\cdot\vec{l})^2 = (\vec{p}\times\vec{l})\cdot(\vec{p}\times\vec{l})}$$

$$(\vec{p}\times\vec{l} + \vec{l}\times\vec{p})_i = \epsilon_{ijk}[p_j, l_k] = \epsilon_{ijk}\epsilon_{kji'}(-i\hbar) p_{i'}$$

$$= \epsilon_{ijk}\epsilon_{ji'k}(-i\hbar) p_{i'} = (\delta_{ij}\delta_{ji'} - \delta_{ii'}\delta_{jj})(-i\hbar) p_{i'} = -i\hbar(p_i - 3p_i) = 2i\hbar p_i$$

$$\Rightarrow \boxed{\vec{p}\times\vec{l} = -\vec{l}\times\vec{p} + 2i\hbar\vec{p}}$$

$$\boxed{(\vec{p}\times\vec{l})\cdot\vec{p} = 2i\hbar p^2}, \qquad \vec{p}\cdot(\vec{p}\times\vec{l}) = 0$$

$$(\vec{p}\times\vec{l})\cdot\vec{r} = \left[-(\vec{l}\times\vec{p}) + 2i\hbar\vec{p}\right]\cdot\vec{r} = -(\vec{l}\times\vec{p})\cdot\vec{r} + 2i\hbar\vec{p}\cdot\vec{r} = l^2 + 2i\hbar\vec{p}\cdot\vec{r}$$

$$\vec{r}\cdot(\vec{p}\times\vec{l}) = l^2$$

$$\frac{\vec{r}}{r}\cdot\vec{p} = \vec{p}\cdot\left(\frac{\vec{r}}{r}\right) + i\hbar\nabla\cdot\left(\frac{\vec{r}}{r}\right) = \vec{p}\cdot\vec{r}/r + i\hbar\left(\frac{3}{r} - \frac{\vec{r}\cdot\vec{r}}{r^3}\right) = \vec{p}\cdot\vec{r}/r + 2i\hbar\frac{1}{r}$$

$$\Rightarrow m^2k^2A^2 = p^2l^2 - (\vec{p}\cdot\vec{l})^2 - \hbar^2p^2 + (mk)^2 + 2\hbar p^2 - mk\left[\frac{2l^2}{r} + \cancel{2i\hbar\frac{\vec{p}\cdot\vec{r}}{r}}\right]$$

$$+ 2im\hbar k\left[\cancel{\frac{\vec{p}\cdot\vec{r}}{r}} + i\hbar\frac{1}{r}\right]$$

$$\vec{p}\cdot\vec{l} = \vec{p}\cdot(\vec{r}\times\vec{p}) = \epsilon_{ijk} p_i r_j p_k = \epsilon_{ijk}\left[r_j p_i p_k + (-i\hbar\delta_{ij}) p_k\right] = 0$$

$$\Rightarrow m^2k^2A^2 = p^2[l^2 + \hbar^2] - 2m\frac{k}{r}[l^2 + \hbar^2] + (mk)^2$$

$$\Rightarrow A^2 = \frac{2m}{m^2k^2}\left(\frac{p^2}{2m} - \frac{\kappa}{r}\right)(l^2 + \hbar^2) + 1 = \frac{2H}{mk^2}(\vec{l}^2 + \hbar^2) + 1$$

⑥ $\boxed{\vec{A}\cdot\vec{l} = \vec{L}\cdot\vec{A} = 0}$

$\vec{r}\cdot\vec{l} = \vec{r}\cdot(\vec{r}\times\vec{p}) = 0$

① $\vec{A} = \frac{1}{mk}(\vec{p}\times\vec{l} - i\hbar\vec{p}) - \vec{r}/r$

$$\vec{A}\cdot\vec{l} = \frac{1}{mk}(\vec{p}\times\vec{l})\cdot\vec{l} - \frac{i\hbar}{mk}\vec{p}\cdot\vec{l} - \frac{\vec{r}}{r}\cdot\vec{l}$$

$$= 0 + 0 + 0 = 0$$

$\vec{l}\cdot\vec{p} = (\vec{r}\times\vec{p})\cdot\vec{p} = 0$

$\vec{l}\cdot\vec{r} = (\vec{r}\times\vec{p})\,\vec{r}$

$= -(\vec{p}\times\vec{r})\cdot\vec{r} = 0$

$$\vec{A} = \frac{1}{mk}(-\vec{l}\times\vec{p} + i\hbar\vec{p}) - \vec{r}/r$$

$$\vec{l}\cdot\vec{A} = \frac{1}{mk}\left[-\vec{l}(\vec{l}\times\vec{p}) + i\hbar\,\vec{l}\cdot\vec{p}\right] - \vec{l}\cdot\vec{r}$$

$$= 0 + 0 + 0 = 0$$

⑦ Define
$$\boxed{\begin{aligned}\vec{I} &= \frac{1}{2}\left[\vec{L} + \sqrt{\frac{-mk}{2E}}\,\vec{A}\right]\\ \vec{K} &= \frac{1}{2}\left[\vec{L} - \sqrt{\frac{-mk}{2E}}\,\vec{A}\right]\end{aligned}}$$

SO(4) decompose into a pair of SO(3).

pay ~~attention~~ to the coefficient $1/2$

due to $\vec{L}\cdot\vec{A} = \vec{A}\cdot\vec{L} = 0 \Rightarrow \vec{I}\cdot\vec{I} = \vec{K}\cdot\vec{K} = \frac{1}{4}\left[\vec{l}^2 + \left(\frac{-mk}{2E}\right)\vec{A}^2\right]$

we can only realize a special class of SO(4) Rep. $I=K$.

due to coefficients $1/2$, $I=K$ can take both integer/half integers.

$\vec{L} = \vec{I} + \vec{K}$

$\vec{A} = \vec{I} - \vec{K}$

thus $\vec{L}$ is still an angular momentum, but $\vec{A}$ is not! Because $I=K$, L can only be integer-valued!

(8) Figure out the spectrum

$$\left(\sqrt{\frac{-mk^2}{2E}}\,\vec{A}\right)^2 + \vec{L}^2 = -(\vec{L}^2 + \hbar^2) + \frac{mk^2}{-2E} + \vec{\ell}^2$$

$$= -\frac{m^2k^2}{2E} - \hbar^2$$

↓ twice of

the SO(4) Casimar define in Page 12.

$$2\left[I(I+1)\hbar^2 + K(K+1)\hbar^2\right] = 4I(I+1)\hbar^2$$

$I = K$

$$\Rightarrow -\frac{mk^2}{2E} = (2I+1)^2\hbar^2 \Rightarrow E = -\frac{mk^2/\hbar^2}{2(2I+1)^2}$$

where $I = \frac{1}{2}, \frac{3}{2}, \cdots$

$2I+1 = 1, 2, 3 \cdots$

$$mk^2/\hbar^2 = \frac{me^4}{\hbar^2} = \frac{e^2}{\hbar^2/me^2} = \frac{e^2}{a} = \text{Rydberg}$$