Lect 16 Discrete symmetry: time-reversal (TR) and parity

§1 Wigner theorem:

Generally speaking, for a transformation R (not necessarily linear), if it does not change the magnitude of the inner product between two arbitrary state vectors $|\psi\rangle$ and $|\phi\rangle$, i.e., $|\langle\psi|\phi\rangle| = |\langle R\psi|R\phi\rangle|$, then R is either a unitary transformation, or, an anti-unitary transformation. (We will omit the proof). For continuous transformation R is unitary (why?).

Anti-unitary transformation R means that: for a super-position between $|\phi_1\rangle, |\phi_2\rangle$,

$$\boxed{R(c_1|\phi_1\rangle + c_2|\phi_2\rangle) = c_1^* R|\phi_1\rangle + c_2^* R|\phi_2\rangle}, \quad \text{or } Rc = c^* R.$$

usually anti-unitary transformation can be expressed as $R = \mathcal{U} K$, where $\mathcal{U}$ is an usual unitary transformation, and K is anti-unitary satisfying $KK = 1$. In the coordinate representation, we choose K as complex conjugation.

$$\boxed{\langle\vec{r}\,|K|\psi\rangle = \langle\vec{r}\,|\psi\rangle^*}$$

Ex: please check that $R^{-1} = K\mathcal{U}^\dagger = K\mathcal{U}^{-1}$, and we can evaluate

$$\langle R\psi|R\phi\rangle = \langle \mathcal{U}K\psi|\mathcal{U}K\phi\rangle = \langle K\psi|K\phi\rangle = \int dr \langle K\psi|r\rangle\langle r|K\phi\rangle$$

$$= \int dr \langle r|K\psi\rangle^* \langle r|K\phi\rangle = \int dr \langle r|\psi\rangle\langle r|\Phi\rangle^* = \int dr \langle\Phi|r\rangle\langle r|\psi\rangle$$

$$\Rightarrow \boxed{\langle R\psi|R\phi\rangle = \langle\Phi|\psi\rangle}$$

Ex: prove that $\langle R^{-1}\psi|R^{-1}\Phi\rangle = \langle\Phi|\psi\rangle = \langle\psi|\Phi\rangle^*$.

For any states $|\psi\rangle$ and $|\psi'\rangle$, and operater O

$$\langle R\psi|O|R\psi'\rangle = \langle R\psi|R\underbrace{R^{-1}OR|\psi'\rangle} = \langle\psi|R^{-1}OR|\psi'\rangle^*$$

If $|\psi\rangle = |\psi'\rangle$, and O is an Hermitian operator $\Rightarrow \langle R\psi|O|R\psi\rangle \geq 0$

$$\Rightarrow \quad \langle R\psi|O|R\psi\rangle = \langle\psi|R^{-1}OR|\psi\rangle.$$

§2. TR transformation

Consider a state vector $|\psi\rangle$, and its TR counter part $|\psi^T\rangle = T|\psi\rangle$, or, equivalently $|\psi\rangle = T^{-1}|\psi^T\rangle$, We assume T and T^{-1} satisfy Wigner theorem. Now we need to determine T is unitary or anti-unitary. We need correspondence principle.

In order to agree with classic mechanics, we need maintain

$$\begin{cases} \langle\psi^T|\vec{r}|\psi^T\rangle = \langle\psi|\vec{r}|\psi\rangle, \quad \langle\psi^T|\vec{P}|\psi^T\rangle = -\langle\psi|\vec{P}|\psi\rangle \\ \langle\psi^T|\vec{L}|\psi^T\rangle = -\langle\psi|\vec{L}|\psi\rangle \end{cases}$$

then

$$\boxed{\langle\psi^T|\vec{r}|\psi^T\rangle = \langle T\psi|\vec{r}|T\psi\rangle = \langle\psi|T^{-1}\vec{r}\,T|\psi\rangle}$$

$T^{-1}\vec{r}\,T$ is a lineary operater since the product of two anti-linear operators is a linear operator. The above relation is valid for any state vector $|\psi\rangle$. It's easy to show for two aribitary state vectors

$$\boxed{\langle\psi_1|T^{-1}\vec{r}\,T|\psi_2\rangle = \langle\psi_1|\vec{r}|\psi_2\rangle, \quad \text{Such that } T^{-1}\vec{r}\,T = \vec{r}}$$

Proof: take $|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle$,

$$\left(\langle\psi_1| + \langle\psi_2|\right)(T^{-1} r T)\left(|\psi_2\rangle + |\psi_1\rangle\right) = \left(\langle\psi_1| + \langle\psi_2|\right) r \left(|\psi_2\rangle + |\psi_1\rangle\right)$$

$$\Rightarrow \langle\psi_1| T^{-1}\vec{r}T|\psi_2\rangle + \langle\psi_2|T^{-1}\vec{r}T|\psi_1\rangle = \langle\psi_1|\vec{r}|\psi_2\rangle + \langle\psi_2|\vec{r}|\psi_1\rangle$$

if we take $|\psi\rangle = |\psi_1\rangle + i|\psi_2\rangle \Rightarrow$

$$\langle\psi_1|T^{-1}\vec{r}T|\psi_2\rangle - \langle\psi_2|T^{-1}\vec{r}T|\psi_1\rangle = \langle\psi_1|\vec{r}|\psi_2\rangle - \langle\psi_2|\vec{r}|\psi_1\rangle$$

$$\Rightarrow \boxed{\langle\psi_1|T^{-1}\vec{r}T|\psi_2\rangle = \langle\psi_1|\vec{r}|\psi_2\rangle}.$$

Similarly, we should have

$$\boxed{\begin{cases} T^{-1}\vec{r}\,T = \vec{r} \\ T^{-1}\vec{p}\,T = -\vec{p} \\ T^{-1}\vec{L}\,T = -\vec{L} \end{cases} \quad \text{also } T^{-1}\vec{S}\,T = -\vec{S}}$$

at operator level

In order to be consistent with these relation, T has to be anti-unitary.

Check the commutation relation $[x,p] = i\hbar$, how does it change under T?

$$T[x,p]T^{-1} = T i\hbar T^{-1}$$

$$(TxT^{-1})(TpT^{-1}) - (TpT^{-1})(TxT^{-1}) = -(xp - px) = -i\hbar$$

$$\Rightarrow \boxed{T i T^{-1} = -i}$$

Ex: From $[L_i, L_j] = i\epsilon_{ijk}L_k$, derive that $T i T^{-1} = -i$.

§3. $T^2 = ?$

Naively, we would expect that after TR transformation twice, the system comes back to itself, thus $T^2 = 1$. But we will see two possibilities.

First, T^2 is a constant.

Proof: we have $T\vec{r}T^{-1}=\vec{r} \Rightarrow T^2\vec{r}T^{-1^2}=\vec{r} \Rightarrow T^2\vec{r}=\vec{r}T^2$

$T\vec{p}T^{-1}=-\vec{p} \Rightarrow T^2\vec{p}T^{-1^2}=\vec{p} \Rightarrow T^2\vec{p}=\vec{p}T^2$

and similary $T^2\vec{L}=\vec{L}T^2$, $T^2\vec{S}=\vec{S}T^2$, $T^2 i = iT^2$.

For any operater $F(r,p,S,i)$, we have $T^2F(r,p,S,i)=F(r,p,S,i)T^2$

$\Rightarrow$ $\boxed{T^2 \text{ is a constant.}}$

Then what's its value? $\boxed{\text{Answer: } T^4=1, \text{ and thus } T^2=\pm 1.}$

Proof. $T^4 = T(T^2)T = (T^2)^* T^2 = T^2 (T^2)^*$.

For any two state vecters $|\psi\rangle$ and $|\phi\rangle$, rember T is anti-unitary, and T^2 is a complex constant $\Rightarrow$

$\langle T\psi|T\phi\rangle = \langle\psi|\phi\rangle^*$, $\langle T^2\psi|T^2\phi\rangle = \langle T\psi|T\phi\rangle^* = \langle\psi|\phi\rangle$

$\langle T^2\psi|T^2\phi\rangle = (T^2)^*T^2\langle\psi|\phi\rangle = \langle\psi|\phi\rangle \Rightarrow (T^2)^*T^2 = T^4 = 1$

§4 The case of $T^2=1$.

For single component system, we can simply define $\psi^T(r)=\psi^*(r)$, or $\langle r|T|\psi\rangle = \langle r|\psi\rangle^*$. Please check that it satisfies:

$$\int d^3r\,(\psi^T(r))^* \begin{pmatrix}\vec{r}\\ \vec{p}\\ \vec{L}\end{pmatrix} \psi^T(r) = \int d^3r\,\psi^*(r)\begin{pmatrix}\vec{r}\\ -\vec{p}\\ -\vec{L}\end{pmatrix}\psi(r).$$

Is there an example of H, that violates TR symmetry?

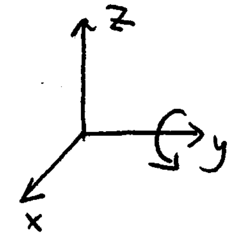
Ex: check: $H = \dfrac{(P-\frac{e}{c}A)^2}{2m}$, what's $H^T = T H T^{-1} = ?$

§5 The case of $T^2=-1$. and Kramer degeneracy.

Let's consider a system with spin, The rotation matrix

$$D(g) = e^{-i\vec{J}\cdot\hat{n}\theta}$$

Let's consider the rotation operation and TR



$$\begin{cases} D^{+}(g(\hat{y},\pi))\, J_z\, D(g(\hat{y},\pi)) = -J_z \\ T^{-1}\, J_z\, T = -J_z \end{cases}$$

$$T^{-1} D^{-1}(g(\hat{y},\pi))\, J_z\, D(g(y,\pi))\, T = J_z$$

or $\boxed{J_z[D(g(y,\pi))\,T] = [D(g(y,\pi))\,T]\, J_z}$

Consider a J_z eigenstate, $|jm\rangle$, then $D(g(y,\pi))\,T\,|jm\rangle$ must be the same as $|m\rangle$ up to a complex constant, because

$$J_z\left(D(g(y,\pi))T|jm\rangle\right) = D(g(y,\pi))\,T\,J_z|jm\rangle = m\left(D(g(y,\pi))\,T|jm\rangle\right)$$

$$\Rightarrow D(g(y,\pi))\,T\,|jm\rangle = c\,|jm\rangle.$$

Then $(D(g(y,\pi))\,T)^2\,|jm\rangle = (D(g(y,\pi))\,T)\,c\,|jm\rangle = c^{*}c\,|jm\rangle.$

$$\langle D(g(y,\pi))\,Tjm|\,D(g(y,\pi))\,Tjm\rangle = \langle Tjm|Tjm\rangle = \langle jm|jm\rangle \Rightarrow c^{*}c = 1$$

Thus $\left(D(g(y,\pi))\, T\right)^2 = 1$ or $T\, D(g(y,\pi))\, T\, D(g(y,\pi)) = 1$

For $T e^{-iJ\cdot\hat{n}\theta} T^{-1} = e^{-(-i)(-\vec{J})\cdot\hat{n}\theta} = e^{-iJ\cdot\hat{n}\theta}$

$\Rightarrow$ $T D(g) = D(g) T$

$\Rightarrow$ $T^2 D^2(g(y,\pi)) = 1$ or $\boxed{T^2 D(g(y,2\pi)) = 1}$

But rotation around y-axis at 2π-angle, should it just an identity transformation? Not guite

$$D(g(y,2\pi)) = \begin{cases} I & j \text{ integer} \\ -I & j \text{ half-integer}. \end{cases}$$

Proof: $D(g(y,2\pi)) = D(g(x,\tfrac{\pi}{2}))\, D(g(z,2\pi))\, D(g^{-1}(x,\tfrac{\pi}{2}))$

$g(x,\frac{\pi}{2})$ rotation rotates y-axis into z-axis

$D(g(z,2\pi)) = e^{-iJ_z 2\pi} = \begin{cases} I & \text{if } J_z \text{ integer} \\ -I & \text{if } J_z \text{ half integer} \end{cases}$

$$\Rightarrow \boxed{D(g(y,2\pi)) = D(g(z,2\pi)) = \begin{cases} I & j \text{ integer} \\ -I & j \text{ half integer}. \end{cases}}$$

$$\Rightarrow \boxed{T^2 = \begin{cases} 1 & \text{for } j \text{ integer} \\ -1 & \text{for } j \text{ half integer} \end{cases}}$$

$\to$ orthogonal class,

$\to$ symplectic class.

1. For spin-$1/2$ case, a convenient choice is $T = -i\sigma_y K = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} K$

$$\begin{cases} T\,|\uparrow\rangle = |\downarrow\rangle \\ T\,|\downarrow\rangle = -|\uparrow\rangle \end{cases} \quad \text{and} \quad T(c_1|\uparrow\rangle + c_2|\downarrow\rangle) = c_1^*|\downarrow\rangle - c_2^*|\uparrow\rangle.$$

K is the complex conjugate on complex coefficent

or $T\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -c_2^* \\ c_1^* \end{pmatrix}$, $T^2\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

2. For a general case, we can define $T = RK$.

① if j half integer, $2j+1$ is even. we may choose $R = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}$.

and $R^2 = -1$, and $T^2 = -1$.

② if j is integer, $2j+1$ is odd. we choose $R = \begin{array}{c} \ m=0 \\ \begin{pmatrix} & & -1 \\ & 1 & \\ -1 & & \end{pmatrix} \end{array} m=0$, such that

$R^2 = T^2 = 1$.

in this case, $T|lm\rangle = (-)^m |l\,-m\rangle$, and

$$\langle \hat{n}|T|lm\rangle = Y_{lm}^*(\theta,\varphi) = (-)^m \langle \hat{n}|l\,-m\rangle = (-)^m Y_{l-m}(\theta,\varphi).$$

consistent with $\boxed{Y_{lm}^*(\theta,\varphi) = (-)^m Y_{l-m}(\theta,\varphi)}$

§ Kramer degeneracy.

if $T^2 = -1$, then for any eigen state $|\psi\rangle$, with $H|\psi\rangle = E|\psi\rangle$

then $H(T|\psi\rangle) = TH|\psi\rangle = E(T|\psi\rangle)$, thus $T|\psi\rangle$ is also an eigenstate with the same energy.

On the other hand

$$\langle\psi|T\psi\rangle = \langle T\psi|T^2\psi\rangle^* = -\langle T\psi|\psi\rangle^* = -\langle\psi|T\psi\rangle$$

$\Rightarrow \langle\psi|T\psi\rangle = 0$. thus $T|\psi\rangle$ is ~~i~~ an other state, and there's at least 2-fold degeneracy.

Ex: if $T^2=1$, is there always an energy level degeracy ?

§ Parity transformation

Consider a state vector $|\psi\rangle$, after parity transformation P, we have $|\psi^P\rangle = \mathbb{P}|\psi\rangle$, or $|\psi\rangle = \mathbb{P}^{-1}|\psi^P\rangle$. Again we assume P satisfies Wigner theorem. Again we use correspondence principle, and arrive at

$$\langle\psi^P|\vec{r}|\psi^P\rangle = -\langle\psi|\vec{r}|\psi\rangle, \quad \langle\psi^P|\vec{P}|\psi^P\rangle = -\langle\psi|\vec{P}|\psi\rangle$$

$\langle\psi^P|\vec{L}|\psi^P\rangle = \langle\psi|\vec{L}|\psi\rangle$, and also $\langle\psi^P|\vec{S}|\psi^P\rangle = -\langle\psi|\vec{S}|\psi\rangle$

$$\Rightarrow \boxed{\mathbb{P}^{-1}\vec{r}\,\mathbb{P} = -\vec{r}, \quad \mathbb{P}^{-1}\vec{P}\,\mathbb{P} = -\vec{P}, \quad \text{and } \mathbb{P}^{-1}\vec{L}\,\mathbb{P} = \vec{L}, \quad \mathbb{P}^{-1}\vec{S}\,\mathbb{P} = \vec{S}}$$

check $[x,p] = i\hbar \Rightarrow \mathbb{P}^{-1}[x,p]\mathbb{P} = [-x,-p] = i\hbar = \mathbb{P}^{-1}(i\hbar)\mathbb{P}$

$\Rightarrow \mathbb{P}i = i\mathbb{P} \Rightarrow$ $\boxed{\mathbb{P} \text{ is an unitary transformation.}}$

Similarly, we can also prove that $\mathbb{P}^2$ is a constant, and $\mathbb{P}^2(\mathbb{P}^2)^* = 1$.

Ex: without loss of generality, we choose

$\Rightarrow \mathbb{P}^2 = e^{i\delta}$ up to phase factor. $\mathbb{P}^2 = 1$.

For single component system, we simply set $\psi^P(\vec{r}) = \psi(-\vec{r})$.

We can easily check this definition satisfy the above requirement!

For the time-dependent case, we can define

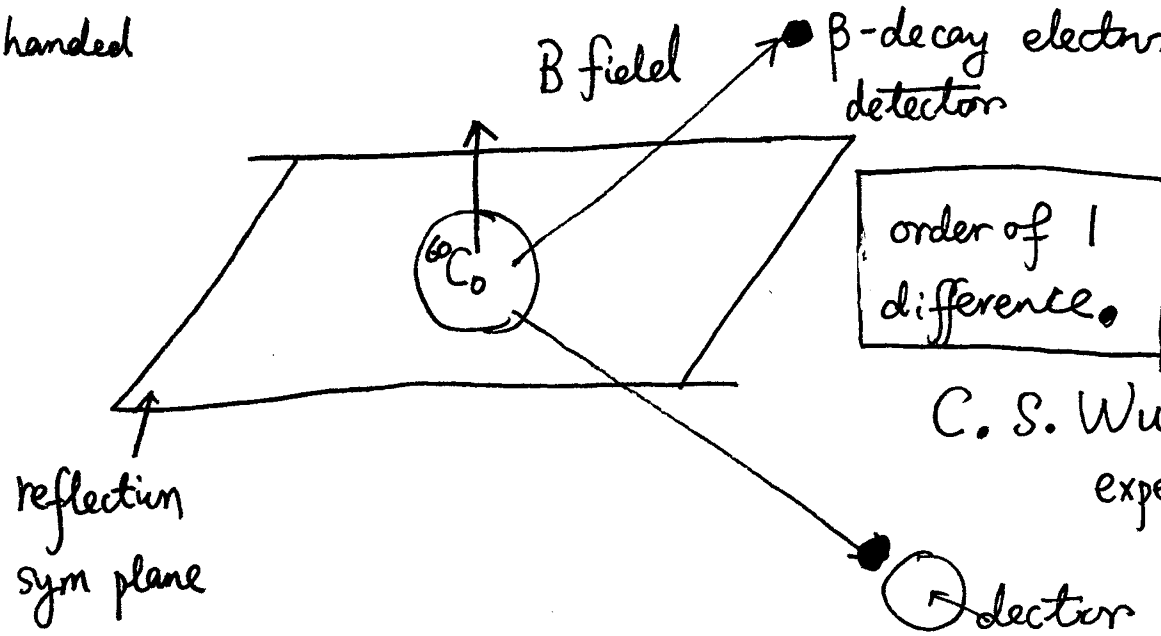
$$\boxed{\begin{aligned} \psi^T(x,t) &= \psi^*(x,-t) \\ \psi^P(x,t) &= \psi(-x,t). \end{aligned}}$$

Ex: verify for momentum eigenstate $\psi_p(x,t) = e^{-ipx-i\omega t}$, what ~~is~~ are $\psi^T_p(x,t)$, and $\psi^P_p(x,t)$? How about angular momentum eigenstates $\psi_m(x,t) = e^{im\varphi - i\omega t}$?

§ Parity broken in weak-interactions. — C.N Yang and T. D. Lee (Theory proposal)

$\begin{pmatrix} e_L \\ \nu_L \end{pmatrix}$, e_R. there's no ν_R.

left handed



C. S. Wu's experiments.

§ Parity eigenstates

If $[H, \mathbb{P}] = 0$, then we can find common eigenstates of H, and $\mathbb{P}$.

For example: ① 1D harmonic oscillator $\mathbb{P}^{-1} H \mathbb{P} = H$. It's energy Wavefun

$$\psi_n(-x) = (-)^n \psi_n(x), \qquad \text{even for } n = 0, 2, 4 \cdots \quad \text{odd for } n = 1, 3, 5 \cdots$$

② orbital angular momentum eigenstates $Y_{lm}(\hat{n})$

$$Y_{lm}(-\hat{n}) = (-)^l Y_{lm}(\hat{n})$$

③ Selection rule $\Delta H = -e\vec{r}\cdot\vec{E}$

$$\langle nlm | \Delta H | n'l'm' \rangle \neq 0, \quad \text{only for } l' = l \pm 1.$$

§ The relation between degeneracy and symmetry.

For a Hamiltonian, all it's symmetry operations R together form a group.

If a state $|\psi\rangle$ is an eigenstate, then all the states $R|\psi\rangle$

$$H(R|\psi\rangle) = R(H|\psi\rangle) = E|\psi\rangle$$

form a degenerate subspace. This subspace support a representation for the symmetry group.

For example: ① for 3D rotation symmetry, all the states $\psi_{nlm} = R_{nl} Y_{lm}(\theta, \varphi)$ with $m = -l, \cdots l$, form a l-fold degeneracy.

② But for 1d harmonic oscillator, the parity symmetry does not bring degeneracy.

whether degeneracy appears or not depends on the nature of the sym group. If the group is Abelian, i.e., every symmetry operation commute with other, we do not expect degeneracy. It's because Abeliean group usually only support 1d representation.

example: ① Parity group $\{I, IP\}$. No degeneray in 1D harmonic oscilator

If we apply $IP\,\psi_n(r) = \pm\,\psi_n(r)$, no new states appear.

② $H = -B\cdot S_z$, uni-axial rotation symmetry.

SO(2) group : $\{e^{-iS_z\theta}\}$. Again for its eigenstates $|S\,S_z\rangle$

we have $\hat{S}_z\,|SS_z\rangle = S_z\,|SS_z\rangle$, no more states.

> Energy level degeneracy are usually associated with non-Abelian sym group. Only non-Abelian group supports multi-dimensional representations

Example: 2d rotator $H = \dfrac{L_z^2}{2I}$.

Each level except the ground state is 2-fold symmtric. $\psi_{\pm m} = e^{\pm im\varphi}$

The symmetry group is O(2) not SO(2): $\{e^{-iL_z\theta}\} \cup \{\Pi_x\, e^{-iL_z\theta}\}$,

where Π_x is the reflection with respect to x-axis. It's easy to check

$$\Pi_x\, e^{-iL_z\theta}\,\Pi_x = e^{iL_z\theta}$$, and thus O(2) is non-abeliean!

For it's eigenstates $\psi_{\pm m}$, we have $\boxed{\begin{array}{l} L_z\,\psi_{\pm m} = \pm m\,\psi_{\pm m}, \text{ and} \\ \Pi_x\,\psi_m = \psi_{-m} \text{ and } \Pi_x\,\psi_{-m} = \psi_m. \end{array}}$