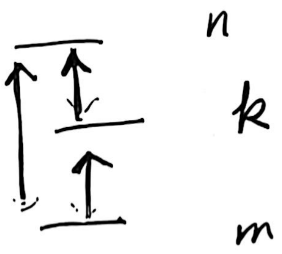
Lecture 2: Heisenberg's magic

§1: Reinterpretation of kinematic variable

$$x_{n,m}\, e^{i\omega(n,m)t}$$



$$= \begin{pmatrix} x_{11} e^{i\omega_{11}t} & x_{12} e^{i\omega_{12}t} & \cdots \\ x_{21} e^{i\omega_{21}t} & x_{22} e^{i\omega_{22}t} & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

$$(x^2)_{nm} = \sum_k (X)_{nk} (X)_{km}$$

$\omega_{nm} = \omega_{nk} + \omega_{km}$

Ritz combination law

$\ddot{x} + f(x) = 0$ ← in the matrix sense

§ Quantization

$\frac{1}{2\pi}\oint p\,dq = n\hbar$ ⟷ $\boxed{\frac{\hbar}{2m} = \sum_{\alpha=0}^{\infty} \left\{ \omega(n+\alpha, n)\, |x(n+\alpha, n)|^2 - \omega(n, n-\alpha)\, |X(n, n-\alpha)|^2 \right\}}$

§ Heinsenberg's zero point motion

$$[(\dot{x})^2]_{nn} = \frac{\hbar\omega}{m}(n+1/2)$$

§ Born - Jordan's matrix mechanics

$[x, p] = i\hbar$ ← Born's tomb stone

Heisenberg's breakthroughs

① Replace the Fourier series of harmonic $X_n(t) = \sum a_\alpha(n) e^{i\alpha\omega_n t}$

with matric-elements, i.e. Heisenberg harmonies

$$X_{n,m} e^{i\omega(n,m)t} \longrightarrow \begin{pmatrix} X_{11}e^{i\omega_{11}t} & X_{12}e^{i\omega_{12}t} & \cdots \\ X_{21}e^{i\omega_{21}t} & X_{22}e^{i\omega_{22}t} & \cdots \\ \vdots & \vdots & \cdots \end{pmatrix}$$

reinterpretation of kinematic coordinate

② how about X^2 ? ← reinterpretation product

$$(X^2)_{nm} e^{i\omega(n,m)t}$$

$$= \sum_k X_{nk} e^{i\omega(n,k)t} X_{km} e^{i\omega(k,m)t}$$

$$= \sum_k X_{nk} X_{km} e^{i\omega(n,m)t}$$ ← Ritz combination law

matrix production

③ reinterpretation – of equation of motion

Heisenberg modified the kinematics, i.e. the definition of X, but he thinks that Hamilton's Eq is still valid, which can still be derived by solving $\ddot{x} + f(x) = 0$. (Maintaining the form of dynamic eq, but reinterprete the Kinematic meaning!)

④ reinterpretation — quantization condition

$$\oint p\,dx = nh \Rightarrow m\oint \dot{x}^2 dt = nh$$

classically $x = \sum_\alpha a_\alpha(n)\, e^{+i\alpha\omega_n t}$, $\rightarrow \dot{x} = \sum_\alpha i\alpha\omega_n a_\alpha(n) e^{i\alpha\omega_n t}$

$$\dot{x}^2 = \sum_\alpha \alpha^2\omega_n^2 |a_\alpha(n)|^2 + \sum_{\alpha\neq\alpha'} \alpha\alpha'\omega_n^2 a_\alpha(n) a^*_{\alpha'}(n) e^{i(\alpha-\alpha')\omega_n t}$$

$$\Rightarrow m\oint \dot{x}^2 dt = m\sum_\alpha \alpha^2\omega_n^2 |a_\alpha(n)|^2 \oint dt \quad \leftarrow T\omega_n = 2\pi$$

$$= \boxed{2\pi m \sum_\alpha \alpha^2 \omega_n |a_\alpha(n)|^2 = nh}$$

how to make it to the transition-type, — take derivation with respect to $n \Rightarrow$

$$\frac{\hbar}{m} = \frac{d}{dn}\sum_\alpha \alpha^2\omega_n |a_\alpha(n)|^2$$

$$= \sum_{\alpha=-\infty}^{+\infty} \alpha \frac{d}{dn}\left[\alpha\omega_n |a_\alpha(n)|^2\right]$$

since $\alpha \ll n$, $\alpha\frac{d}{dn} \rightarrow$ difference

$$\omega(n+\alpha, n)|X(n+\alpha,n)|^2 - \omega(n,n-\alpha)|X(n,n-\alpha)|^2$$

$$\Rightarrow \frac{\hbar}{m} = \sum_{\alpha=-\infty}^{\infty} \omega(n+\alpha,n)|X(n+\alpha,n)|^2 - \omega(n,n-\alpha)|X(n,n-\alpha)|^2$$

since $\omega(n-\alpha,n) = -\omega(n,n-\alpha)$, $|X(n-\alpha,n)|^2 = |X(n,n-\alpha)|^2$

also $\omega(n,n)=0$, we can simplify to only including $\alpha>0$

$$\boxed{\frac{\hbar}{2m} = \sum_{\alpha=0}^{\infty}\left\{\omega(n+\alpha,n)|X(n+\alpha,n)|^2 - \omega(n,n-\alpha)|X(n,n-\alpha)|^2\right\}} \leftarrow \text{reinteperated Quantum condition.}$$

(*) Application — harmonic oscillators

$$\ddot{x} + \omega_0^2 x = 0$$

plug in $X(n\pm\alpha, n) \quad e^{i\omega(n\pm\alpha, n)t}$

$$\Rightarrow (-\omega^2(n\pm\alpha, n) + \omega_0^2)\, X(n\pm\alpha, n)\, e^{i\omega(n\pm\alpha, n)t} = 0$$

① if $\alpha = 0$, since $\omega(n,n) = 0$, $\Rightarrow X(n,n) = 0$

unique

② We assume for each n, there should exist an α, such that $X(n+\alpha, n)$, or $X(n-\alpha, n) \neq 0$. Otherwise, such an "n"-state is unobservable!

If there exist α, β, such that $X(n+\alpha, n) \neq 0$, $X(n+\beta, n) \neq 0$

$\omega(n+\alpha, n) = \omega_0 = \omega(n+\beta, n)$

$$\Rightarrow \omega(n+\alpha, n+\beta) = \omega(n+\alpha, n) + \omega(n, n+\beta) = \omega_0 - \omega_0 = 0$$

$\Rightarrow \alpha = \beta$. (1D motion, no-degeneracy)

③ As $n \gg 1$, we have $\begin{cases} \omega(n+1, n) = \omega_n = \omega_0 \\ \omega(n, n-1) = \omega_n = \omega_0 \end{cases}$

Since harmonic oscillator ω_0 is independent on amplitude, ie, independent on n, we think $\omega(n+1, n) = \omega(n, n-1) = \omega_0$ should be preserved down to $n = 0$. Hence

$$\begin{cases} \omega(n\pm 1, n) = \omega_0 \\ X(n\pm 1, n) \neq 0 \end{cases}$$

if $n = 0$, then $n-1$ is not allowed.

then plug in $\frac{\hbar}{2m} = \omega(n+1,n)\,|X(n+1,n)|^2 - \omega(n,n-1)\,|X(n,n-1)|^2$

$= \omega_0 \left[|X(n+1,n)|^2 - |X(n,n-1)|^2 \right]$

If $n=0 \Rightarrow \sqrt{\frac{\hbar}{2m\omega_0}} = X(1,0)$ assume $|X(n,n-1)|$ does not exist

$n=1$ $\quad \frac{\hbar}{2m\omega_0} = |X(2,1)|^2 - |X(1,0)|^2$

$\vdots \Rightarrow |X(n+1,n)|^2 = n\frac{\hbar}{2m\omega_0} \Rightarrow$ $\boxed{\begin{array}{l} X(n+1,n) = \sqrt{\frac{(n+1)\hbar}{2m\omega_0}} \\ X(n-1,n) = \sqrt{\frac{n\hbar}{2m\omega_0}} \end{array}}$

(*) Now we check $\oint p\,dq = m\int_0^T \dot{x}\dot{x}\,dt$

$(\dot{x}\dot{x})_{nn} = \sum \dot{X}_{nm}\dot{X}_{mn} = \dot{X}_{n,n-1}\dot{X}_{n-1,n} + \dot{X}_{n,n+1}\dot{X}_{n+1,n}$

$\dot{X}_{n,n-1} = \left(\frac{n\hbar}{2m\omega_0}\right)^{1/2} e^{-i\omega_0 t}(-i\omega_0)$

$\Rightarrow (\dot{x}\dot{x})_{nn} = \omega_0^2\left[\frac{n\hbar}{2m\omega_0} + \frac{(n+1)\hbar}{2m\omega_0}\right] = \frac{\hbar\omega_0}{m}(n+1/2)$

$\left(\oint p\,dq\right)_{nn} = \hbar\omega_0(n+1/2)\int_0^T dt = \hbar\omega_0 T\,(n+1/2)$

$$\boxed{\left(\oint \frac{p\,dq}{2\pi}\right)_{nn} = (n+1/2)\hbar = J}$$

$\Rightarrow$ Since $\omega = \frac{\partial E}{\partial J} \Rightarrow E = (n+1/2)\hbar\omega$

$1/2$ gives the zero point motion!

* Born-Jordan commutation relation!

Since X, P, now do not commute, it should be important to evalute $XP - PX = m(X\dot{X} - \dot{X}X)$, check diagonal part $(X\dot{X} - \dot{X}X)_{nn}$

$$\sum_{\alpha}\left[X(n,n-\alpha)\,\dot{X}(n-\alpha,n) - \dot{X}(n,n-\alpha)\,X(n-\alpha,n)\right]$$

$$= \sum_{\alpha}\left[i\,\omega(n-\alpha,n)\,|X(n,n-\alpha)|^2 - i\,\omega(n,n-\alpha)\,|X(n-\alpha,n)|^2\right]$$

if $\alpha > 0$, $\omega(n-\alpha,n) = -\omega(n,n-\alpha)$, $|X(n,n-\alpha)|^2 = |X(n-\alpha,n)|^2$

$$\rightarrow \sum_{\alpha>0} -2i\,\omega(n,n-\alpha)\,|X(n,n-\alpha)|^2$$

if $\alpha < 0$ $\quad 2i\omega(n+|\alpha|,n)\,|X(n,n+|\alpha|)|^2 = 2i\omega(n+|\alpha|,n)\,|X(n+|\alpha|,n)|^2$

$$\Rightarrow (X\dot{X} - \dot{X}X)_{nn} = 2i\sum_{\alpha>0}\left\{\omega(n+\alpha,n)\,|X(n+\alpha,n)|^2 - \omega(n,n-\alpha)\,|X(n,n-\alpha)|^2\right\}$$

$$= \frac{2i\hbar}{2m}$$

$$\Rightarrow \boxed{(XP - PX)_{nn} = i\hbar}$$

⊛ Bohr and Jordan's work

• Postulate 1: q and p are represented as matrices

$q_{mn}\, e^{i\omega(m,n)t}$ and $P_{mn}\, e^{i\omega(m,n)t}$.

$\omega(m,n)$ frequencies associated with the transitions from $m \to n$.

$q(mn)\, q(nm) = |q(nm)|^2$ and $\omega(m,n) = -\omega(n,m)$.

q and p are Hermitian matrices

$|q(nm)|^2$ is a measure of transition probability $n \leftrightarrow m$.

• Postulate 2: frequency recombination

$\omega(j\,k) + \omega(k\,l) + \omega(l,j) = 0.$ ← Ritz combination

At this stage, we can only relate $\hbar\omega(m,n) = W_m - W_n$,

but whether W is energy or not, we still need wait to prove.

Ritz rule assures that any quantity $g(p,q)$, when expressed in terms of the matrix form $g_{mn}\, e^{iW_{mn}t}$, the oscillation frequency W_{mn} is the same. $e^{iW_{mn}t}$ is the universal time factor for all mechanical variables. This ~~because~~ of when p and q multiply, they follow the matrice product law

$$(g_1 g_2)_{mn} \to \sum_k g_{mk}\, e^{iW_{mk}t}\, g_{kn}\, e^{iW_{kn}t}$$

$$= \left(\sum_k g_{mk}\, g_{kn}\right) e^{iW_{mn}t}$$

• Postulate 3. Hamiltonian Eq is the same as before.

$$\begin{cases} H = \dfrac{p^2}{2m} + U(q) \\ \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases}$$

, but should be intepreted in terms of matrix language.

Dynamics rules are the same, but the interpretation of kinematics (meaning of q and p) needs to be redone.

• Postulate 4. The diagonal elements $H(n,n)$ of the Hamiltonian are interpreted as the energy of the state n.

In the old theory, it is difficult to explain why energy is discrete. Now energy appears as the eigenvalues of a Hermitian matrix, hence its discrete nature is explicit.

• Postulate 5: Quantum condition.

$$(pq - qp)_{nn} = \hbar / i$$

⊛ Proof of the commutation relation

Born guessed and postulate $(qp-pq)_{nn} = i\hbar$, but it's different from $[q,p]=i\hbar$ since we have prove that the off diagonal elements vanishes.

Define $g = qp - pq$, and $\dot{g}_{mn} = i\omega(m,n)\, g(m,n)\, e^{i\omega(m,n)t}$.

If $\dot{g}_{mn} = 0$, for $m \neq n$, $\omega(m,n) \neq 0 \Rightarrow g(m,n) = 0$.

for $m = n$, $\omega(m,n) = 0$, $g(m,n)$ may be non zero.

$$\frac{dg}{dt} = \dot{q}p + q\dot{p} - \dot{p}q - p\dot{q} = \frac{\partial H}{\partial p} p - p\frac{\partial H}{\partial p} - q\frac{\partial H}{\partial q} + \frac{\partial H}{\partial q} q$$

① If $H(p,q) = H(p) + H(q)$ which $\overset{\frown}{\text{in}}$ p and q are separatable, then

$dg/dt = 0$ is obervious

② If $p^n q$ type $\overset{\Leftrightarrow}{}$ $qp^n + pqp^{n-1} + \cdots p^m q p^{n-m} + p^n q$

$$\frac{\partial H}{\partial p} = nqp^{n-1} + qp^{n-1} + pqq^{n-2}(n-1) + \cdots mp^{m-1}qp^{n-m} + (n-m)p^m q p^{n-m-1} + np^{n-1}q$$

$$= \sum_m m p^{m-1} q p^{n-m} + (n-m) p^m q p^{n-m-1}$$

$$\rightarrow (n-m+1)\, p^{m-1} q p^{n-m}$$

$$= \sum_m (n+1)\, p^{m-1} q p^{n-m}$$

$$\frac{\partial H}{\partial q} = (n+1)p^n \quad \Rightarrow \quad \frac{dg}{dt} = (n+1)\sum_m \left(p^{m-1} q p^{n-m}\right)p - (n+1)\sum_m p^m q p^{n-m}$$

$$- (n+1)\, qp^n + (n+1)\, p^n q$$

$$= (n+1)\left[qp^n - p^n q - qp^n + p^n q\right] = 0$$

③ For a general term $H = \{p^n, q^m\}$, which means a fully symmetrize combination. There should be $\binom{m+n}{m}$ terms, i.e. $\sum \underbrace{\bullet \circ \bullet \cdots \circ}_{m+n}$, $\bullet$ p, $\circ$ q

$$\frac{\partial}{\partial p}\left(\sum \bullet \circ \circ \bullet \right) = (m+n) \sum \underbrace{\bullet \circ \cdots}_{m+n-1}, \ \{p^{n-1}, q^m\}$$

$\binom{m+n}{n}$ term, each term generats n term of p^{n-1}, q^m

$$n\binom{m+n}{n} = \frac{(m+n)!}{m!\,(n-1)!} = (m+n)\binom{m+n-1}{n-1}$$

$$\Rightarrow \frac{dg}{dt} = (m+n)\Big(\{p^{n-1}, q^m\}p - p\{p^{n-1}, q^m\} - q\{p^n, q^{m-1}\} + \{p^n, q^{m-1}\}q\Big)$$

$$\{p^{n-1}, q^m\}p + \{p^n, q^{m-1}\}q = \{p^n, q^m\}$$

$$\binom{m+n-1}{m} + \binom{m+n-1}{n} = \frac{(m+n-1)!}{m!\,(n-1)!} + \frac{(m+n-1)!}{n!\,(m-1)!}$$

$$= \frac{(m+n-1)!}{(m-1)!\,(n-1)!}\left[\frac{1}{m} + \frac{1}{n}\right] = \frac{(m+n)(m+n-1)!}{mn\,(m-1)!\,(n-1)!} = \binom{m+n}{m}$$

Check all the distributions of p & q

Similarly $p\{p^{n-1}, q^m\} + q\{p^n, q^{m-1}\} = \{p^n, q^m\}$

$$\Rightarrow \boxed{\frac{dg}{dt} = 0!}$$

$$\boxed{\text{Equation of motion:}\quad \dot{g} = \frac{1}{i\hbar}[g, H].}$$

Proof: Check $\dot{q}$ and $\dot{p}$ first. Assume that $H = H_1(p) + H_2(q)$, in which p and q are separatable. According to Hamiltonian Eq.

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{\partial H_1(p)}{\partial p} \qquad \text{Assume } H_1(p) = \sum_n a_n p^n$$

pick up one term p^n, $\Rightarrow \frac{\partial H_1}{\partial p} = n p^{n-1}$

then we can chen $[q, p^n] = qp^n - p^n q = qp^n - pq\cdot p^{n-1} + pq\cdot p^{n-1} - p^2 q p^{n-2} + \cdots p^{n-1} q p - p^n q$

$$= [q,p]p^{n-1} + p[q,p]p^{n-2} + \cdots p^{n-1}[q,p]$$

$$\boxed{[q, p^n] = i\hbar\, n\, p^{n-1}}$$

$$\Rightarrow \frac{\partial p^n}{\partial p} = \frac{1}{i\hbar}[q, p^n] \Rightarrow \frac{\partial \sum a_n p^n}{\partial p} = \frac{1}{i\hbar}[q, \sum a_n p^n]$$

$$\Rightarrow \dot{q} = \frac{\partial H}{\partial p} = \frac{1}{i\hbar}[q, H]$$

Similarly $[p, q^n] = -i\hbar n q^{n-1} \Rightarrow \boxed{\dot{p} = -\frac{\partial H_2}{\partial q} = \frac{1}{i\hbar}[p, H]}$

These relations make the correspondence between Poisson bracket and $\{\ \}_{cl} \longleftrightarrow \frac{1}{i\hbar}[\ ]$.

If $\dot{g}_1 = \frac{1}{i\hbar}[g_1, H]$, $\dot{g}_2 = \frac{1}{i\hbar}[g_2, H]$, then how about $(\dot{g_1 g_2})$

$$\frac{d}{dt}(g_1 g_2) = \dot{g}_1 g_2 + g_1 \dot{g}_2 = \frac{1}{i\hbar}\left([g_1, H]g_2 + g_1[g_2, H]\right) = \frac{1}{i\hbar}[g_1 g_2, H]$$

Hence for $g = \sum a_{mn} p^m q^n$, we have $\boxed{\frac{d}{dt} g = \frac{1}{i\hbar}[g, H].}$

(*) Energy conservation theorem

$$\begin{cases} \dot{H} = 0 \\ \hbar\omega(m,n) = H(mm) - H(nn) \end{cases}$$

Proof

① Although p and q depend on t, $H(p(t), q(t))$ is time-invariant.

$\dot{H} = \frac{1}{i\hbar}[H, H] = 0.$ $\Rightarrow$ $H_{mn} e^{i\omega_{mn}t}$, hence H has to be diagonal otherwise, its off-diagonal term is tim-dependent

② Actually it is not obvious to relate $\omega(m,n)$ with the diagonal elements of H. $\omega(m,n)$ is the transition quantity.

is In Bohr's old quantum theory, $E_m - E_n = \hbar\omega_{mn}$ is just a postulate.

From $\dot{q} = \frac{1}{i\hbar}[q, H] \Rightarrow i\omega_{mn} q_{mn} = \frac{1}{i\hbar}\sum_k q_{mk} H_{kn} - H_{mk} q_{kn}$

$$\hbar\omega_{mn} q_{mn} = H_{mm} q_{mn} - q_{mn} H_{nn} = (H_{mm} - H_{nn}) q_{mn}$$

Similarly $\hbar\omega_{mn} O_{mn} = (H_{mn} - H_{nn}) O_{mn}$ for any quantity O

$\Rightarrow \boxed{\hbar\omega_{mn} = H_{mm} - H_{nn}}$ proof to Bohr's postulate of frequency laws.

$$\Rightarrow O : O_{mn} e^{i(H_{mm} - H_{nn})t/\hbar}$$

Hence, the energy difference between m and n states is the driving force of time-evolution!

(Dirac's paper)

$$\circledast \quad \frac{1}{i\hbar}[\quad] \longleftrightarrow \{\quad\}_{\text{poisson}}$$

Dirac further pointed out the classic limit of Commutator.

$X(n, n-\alpha)$ is considered in the limit of large values of n.

And n refers to the index of action $\oint p\,dq = (n + \text{const})h$.

Consider $(Xp - pX)_{n,n-\gamma}$ ← assume its depedence on n is slow

$$= \sum_{\alpha} X(n, n-\alpha)\, P(n-\alpha, n-\gamma) - P(n, n-(\gamma-\alpha))\, X(n-(\gamma-\alpha), n-\gamma)$$

$$= \sum_{\alpha} \left(X_{n,n-\alpha} - X_{n-(\gamma-\alpha),\, n-\gamma}\right) P_{n-\alpha, n-\gamma} - \left(P_{n,\, n-(\gamma-\alpha)} - P_{n-\alpha, n-\gamma}\right) X_{n-(\gamma-\alpha), n-\gamma}$$

(matrix elements, whose orders switchable)

$$X_{n,n-\alpha} - X_{n-(\gamma-\alpha),\, n-\gamma} = (\gamma-\alpha)\frac{\partial}{\partial n} X_{n,n-\alpha}$$

$$P_{n,n-(\gamma-\alpha)} - P_{n-\alpha, n-\gamma} = \alpha \frac{\partial}{\partial n} P_{n,\, n-(\gamma-\alpha)}$$

$$\Rightarrow (Xp - pX)_{n,n-\gamma} = \sum_{\alpha} \hbar(\gamma-\alpha)\left(\frac{\partial}{\partial n\hbar} X_{n,n-\alpha}\right) P_{n-\alpha, n-\gamma}$$

$$- \hbar\alpha\left(\frac{\partial}{\partial (n\hbar)} P_{n,\, n-(\gamma-\alpha)}\right) X_{n-(\gamma-\alpha),\, n-\gamma}$$

Now restore time-dependence

$\theta = \omega t$ (classic fundamental freq.)

$$(XP - PX)_{n,n-\gamma}\, e^{i\omega_{n,n-\gamma} t}$$

$\omega_{n-\alpha,n-\gamma} t \to (\gamma-\alpha)\omega t = (\gamma-\alpha)\theta$

$$= \frac{\hbar}{i} \sum_\alpha \frac{\partial}{\partial J} X_{n,n-\alpha}\, e^{i\omega_{n,n-\alpha} t} \cdot \frac{\partial}{\partial\theta} P_{n-\alpha,n-\gamma}\, e^{i\omega_{n-\alpha,n-\gamma} t}$$

$$- \frac{\partial}{\partial J} P_{n,n-(\gamma-\alpha)}\, e^{i\omega_{n,n-(\gamma-\alpha)} t} \frac{\partial}{\partial\theta} X_{n-(\gamma-\alpha),n-\gamma}\, e^{i\omega_{n-(\gamma-\alpha),n-\gamma}\cdot t}$$

set $\gamma-\alpha \to \alpha$

$$= \frac{\hbar}{i} \sum_\alpha \left(\frac{\partial}{\partial J} X_{n,n-\alpha}\, e^{i\omega_{n,n-\alpha} t}\right)\left(\frac{\partial}{\partial\theta} P_{n-\alpha,n-\gamma}\, e^{i\omega_{n-\alpha,n-\gamma} t}\right)$$

$$- \left(\frac{\partial}{\partial J} P_{n,n-\alpha}\, e^{i\omega_{n,n-\alpha} t}\right)\left(\frac{\partial}{\partial\theta} X_{n-\alpha,n-\gamma}\, e^{i\omega_{n-\alpha,n-\gamma} t}\right)$$

$$= \frac{\hbar}{i}\left[\frac{\partial}{\partial J} X \frac{\partial}{\partial\theta} P - \frac{\partial}{\partial J} P \frac{\partial}{\partial\theta} X\right]$$

$$= i\hbar\left[\frac{\partial}{\partial\theta} X \frac{\partial}{\partial J} P - \frac{\partial}{\partial\theta} P \frac{\partial}{\partial J} X\right] = i\hbar\{X, P\}$$

Hence, we arrive at the commutator $\leftrightarrow$ Poisson bracket in terms of $\{\theta, J\}$, i.e. phase angle and action.

(3)

$$\frac{\partial}{\partial \theta} f_1 \frac{\partial}{\partial J} f_2 - \frac{\partial f_2}{\partial \theta}\frac{\partial f_1}{\partial J}$$

$$= \left(\frac{\partial}{\partial q} f_1 \frac{\partial q}{\partial \theta} + \frac{\partial f_1}{\partial p}\frac{\partial p}{\partial \theta}\right)\left(\frac{\partial}{\partial q} f_2 \frac{\partial q}{\partial J} + \frac{\partial f_2}{\partial p}\frac{\partial p}{\partial J}\right)$$

$$- \left(\frac{\partial}{\partial q} f_2 \frac{\partial q}{\partial \theta} + \frac{\partial f_2}{\partial p}\frac{\partial p}{\partial \theta}\right)\left(\frac{\partial}{\partial q} f_1 \frac{\partial q}{\partial J} + \frac{\partial f_1}{\partial p}\frac{\partial p}{\partial J}\right)$$

$$= \left(\frac{\partial f_1}{\partial q}\frac{\partial f_2}{\partial p}\left[\frac{\partial q}{\partial \theta}\frac{\partial p}{\partial J} - \frac{\partial q}{\partial J}\frac{\partial p}{\partial \theta}\right]\right.$$

$$\left. - \frac{\partial f_2}{\partial q}\frac{\partial f_1}{\partial p}\right)$$

Hence $\{\ \}_{\theta, J} = \{\ \}_{q, p} \dfrac{\partial(q, p)}{\partial(\theta, J)}$ ← Jacobian.

The cannoical transformation from (q, p) to (θ, J), the generation function is $S_0(q, J)$

$$\left\{ \begin{array}{l} p = \left(\dfrac{\partial S_0}{\partial q}\right)_E = \left(\dfrac{\partial S_0}{\partial q}\right)_J \\ \theta = \left(\dfrac{\partial S_0}{\partial J}\right)_q \end{array} \right. , \quad H = H'$$

$$p\, dq - H\, dt = \overset{-\theta dJ}{} - H' dt + d S_0(q, J)$$

$$\Rightarrow \cancel{\left(\frac{\partial(q, p)}{\partial(\theta, J)}\right)^{-1} = \frac{\partial(q, p)}{\partial(\theta, p)} \Big/ \frac{\partial(\theta, p)}{\partial(\theta, J)}}$$

$$\frac{\partial(\theta, J)}{\partial(q, p)} = \frac{\partial(\theta, J)}{\partial(q, J)} \cancel{/} \cdot \frac{\partial(q, J)}{\partial(q, p)} = \frac{\partial(\theta, J)}{\partial(q, J)} \Big/ \frac{\partial(q, p)}{\partial(q, J)}$$

numeraiter $\left(\frac{\partial \theta}{\partial q}\right)_J = \left(\frac{\partial}{\partial q}\left(\frac{\partial}{\partial J} S_0\right)_q\right)_J$

denominator $\left(\frac{\partial P}{\partial J}\right)_q = \left(\frac{\partial P}{\partial J}\left(\frac{\partial}{\partial q} S_0\right)_J\right)_q$

$$\Rightarrow \quad \frac{\partial(\theta, J)}{\partial(q, p)} = 1$$

The possion bracket's definition

is invariant under canonic transformation.