Lecture 3 The birth of Matrix mechnics

Born, Jordan 1925

Dirac, P A M 1925

$[p, x] = \frac{\hbar}{i}$ ← Born's tombstone

$$\left.\begin{aligned} \dot{x} &= \frac{1}{i\hbar}[x, H] \\ \dot{p} &= \frac{1}{i\hbar}[p, H] \end{aligned}\right\}$$ Born and Jordan

$$\frac{1}{i\hbar}[x, p] \leftrightarrow \{x, p\}$$

$\frac{1}{i\hbar}[f_1, f_2] \leftrightarrow \{f_1, f_2\}$ ← Dirac canonical quantization

§1 Hermitian matrix $\longleftrightarrow$ mechanical quantity

> **Post 1:** Coordinate and momentum should be represented as matrix $p_{m,n}\, e^{-i\omega_{m\leftarrow n}t}$, $x_{mn}\, e^{-i\omega_{m\leftarrow n}t}$.

When we define matrix product, say $(x\,p)_{mn}\, e^{-i\omega_{m\leftarrow n}t}$

$$= \sum_k x_{mk}\, p_{kn}\, e^{-i(\omega_{m\leftarrow k}+\omega_{k\leftarrow n})t}$$

> Hence, Post 1 really relys on the Ritz combination rule.

(x, p) forms the complete set of mechanical observables, $O(p,x)$ in principle can be expressed power series of x and p. We write down the classic expression

$$O = \sum_l O_n(l)\, e^{-il\omega_n t}.$$

The correspondence between classic and quantum version

$$O_n(l)\, e^{-il\omega_n t} \longleftrightarrow O_{n-l,n}\, e^{-i\omega_{n-l\leftarrow n}t}$$ 向下兼容

for l>0

注意规定

$$\text{but } O_{n+l,n}\, e^{il\omega_{n\leftarrow n+l}t}$$
$$= O^*_{n,n+l}\left(e^{-il\omega_{n\leftarrow n+l}t}\right)^*$$

Post 2. frequency recombination

$$\omega_{j\leftarrow k} + \omega_{k\leftarrow l} + \omega_{l\leftarrow j} = 0$$

At this stage, we can relate $\hbar\omega_{m\leftarrow n} = W_n - W_m$.
But whether W is energy or not, we will need to prove.

Post 3. Hamiltonian Eq is the same as before, but should be intepreted in terms of the matrix language.

$$H = \frac{p^2}{2m} + V(q)$$

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

Dynamical rules are the same but the interpretation of kinematis (meaning of q and p) needs to be redone.

§ Establishment of the canonical commutation rule

Post 4: The diagonal matrix elements of

$$(px - xp)_{nn} = \hbar/i .$$

we cannot prove, but show its equivalente to the f-sum rule.

$$(px - xp)_{nn} = m(\dot{x}x - x\dot{x})_{nn} = m\sum_k \dot{x}_{nk} x_{kn} - x_{nk}\dot{x}_{kn}$$

$$\dot{x}_{nk} = -i\omega_{n\leftarrow k} x_{nk} , \quad \dot{x}_{kn} = -i\omega_{k\leftarrow n} x_{kn} .$$

$$(px - xp)_{nn} = -i\sum_k \omega_{n\leftarrow k} x_{nk} x_{kn} - \omega_{k\leftarrow n} x_{nk} x_{kn}$$

① Set $k = n+l$, for $l > 0$

$$m\sum_{l>0} -i\omega_{n\leftarrow n+l} |x_{n,n+l}|^2 + i\omega_{n+l\leftarrow n} |x_{n+l,n}|^2$$

$$= -2mi\sum_{l>0} \omega_{n\leftarrow n+l} |x_{n+l,n}|^2$$

② if $l < 0$, then $k = n - |l|$,

$$(px - xp)_{nn} = -2im\sum \omega_{n\leftarrow n-|l|} x_{n,n-|l|} , x_{n-|l|,n}$$

$$= 2im \sum \omega_{n-|l|,n} |X_{n,n-|l|}|^2$$

$$\Rightarrow (px - xp)_{nn} = 2im \sum_{l>0} \omega_{n\leftarrow n+l} \left[|X_{n+l,n}|^2 - \omega_{n-l\leftarrow n} |X_{n,n-l}|^2 \right]$$

by using the f-sum rule

$$(px - xp)_{nn} = 2im \frac{\hbar}{2m} = \hbar/i .$$

Hence, we show that Heisenberg's quantization rule is equivalent to the diagonal matrix element $[p, x] = \hbar/i$.

* Jordan's contribution: proof of $([p.x])_{m,n} = 0$ if $m \neq n$.

Basically, Jodan proved $\frac{d}{dt}([p,x])_{m,n} = 0$, if the off diagonal matrix element is non-zero, then it would have time dependence $[p,x]_{mn} e^{-i\omega_{mn}t}$.

We consider a simple case that $H = H_1(p) + H_2(x)$

Define $g = \frac{i}{\hbar}[p, x]$.

Then $$\frac{dg}{dt} = \frac{i}{P}\left[\dot{p}x + p\dot{x} - \dot{x}p - x\dot{p}\right]$$

$$\dot{g} = \frac{i}{\hbar}\left(- \frac{\partial H_2(x)}{\partial x} x + x \frac{\partial H_2(x)}{\partial x} - \frac{\partial H_1(p)}{\partial p} p + p \frac{\partial H_1(p)}{\partial p} \right)$$ ⑤

Since H_2 only depends on x, we have $x \frac{\partial H_2(x)}{\partial x} = \frac{\partial H_2(x)}{\partial x} x$,

then $\dot{g} = 0$, then the off-diagonal matrix element

of $g_{m,n} = 0 \Rightarrow g = 1$, i.e $\boxed{[p, x] = \hbar / i}$

If H contains term mixing p, x, the proof is more complicated. Nevertheless, it can be done. We will not show the details here.

⊛ With cannoical commutation rule, we can express

$$\dot{p} = \frac{1}{i\hbar}[p, H], \quad \dot{x} = \frac{1}{i\hbar}[x, H]$$

Again for simplicity, we prove the case $H = H_1(p) + H_2(x)$.

Check $\dot{p} = -\frac{\partial}{\partial x} H_2(x)$, assume $H_2(x) = \sum_n a_n x^n$

$$\Rightarrow \dot{p} = -\sum_n a_n n x^{n-1}.$$

It's easy to show $[p, x^n] = \sum_{i=0}^{n-1} x^i [p, x] x^{n-i-1}$

$$= -i\hbar x^{n-1}$$

$$\Rightarrow \dot{p} = \frac{1}{i\hbar}[p, H].$$

Similarly, we have $\dot{x} = \frac{1}{i\hbar}[x, H]$.

If $\dot{g}_1 = \frac{1}{i\hbar}[g_1, H]$, $\dot{g}_2 = \frac{1}{i\hbar}[g_2 H]$, then

$$\frac{d}{dt}(g_1 g_2) = \dot{g}_1 g_2 + g_1 \dot{g}_2 = \frac{1}{i\hbar}\left([g_1 H] g_2 + g_1 [g_2 H]\right)$$

$$= \frac{1}{i\hbar}[g_1 g_2, H]$$

Since O in principle could be expanded in terms of power series of p, x, then $\dot{O} = \frac{1}{i\hbar}[O, H]$.

⊛ if $H(p,x)$ doesnot depend on t explicitly, then

$\frac{d}{dt} H = 0$. $\Rightarrow$ H is diagonal since its off diagonal matrix elements are time-depedent if it is non-zero.

Post 5: $H_{nn} = E_n$, i.e the diagonal matrix elements of H is the energy of the stationary state E_n.

$\because$ $\dot{O} = \frac{1}{i\hbar}[O, H]$, $\dot{O}_{mn} = \frac{1}{i\hbar}(O_{mn} H_{nn} - H_{mm} O_{mn})$

$$-i\omega_{m\leftarrow n} O_{mn} = \frac{1}{i\hbar} O_{mn} (E_n - E_m)$$

$$\boxed{\hbar \omega_{m\leftarrow n} = E_n - E_m}$$

§ Canonical quantization

$$\{f_1, f_2\}_{p,x} = -\frac{\partial f_1}{\partial p}\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial p}$$

Poisson bracket doesnot depend on the choice of coordinate and momentum. We can also use (J, θ) to define Poisson bracket $\{f_1 f_2\} = \{f_1\ f_2\}_{p,x} = \{f_1 f_2\}_{J,\theta} = -\frac{\partial f_1}{\partial J}\frac{\partial f_2}{\partial \theta} + \frac{\partial f_1}{\partial \theta}\frac{\partial f_2}{\partial J}$

Classical mechanics

$$\dot{p} = \{p, H\}$$
$$\dot{q} = \{q, H\}$$

QM

$$\dot{p} = \frac{1}{i\hbar}[p, H]$$
$$\dot{q} = \frac{1}{i\hbar}[q, H]$$

$\rightarrow$ conjecture $\{\ \} \longrightarrow \frac{1}{i\hbar}[\ ]$.

We will use coordinate and momentum to check the correspondence between $\{\ \}$ and $\frac{1}{i\hbar}[\ ]$.

Consider the matrix element $[p, x]_{n,n-l}$ where n correspondes to the classic action $J = n\hbar$. The classic angular variable $\theta = \omega_n t$ and $\omega_n = \frac{\partial E}{\partial J}\Big|_{J=n\hbar}$. At $n \to +\infty$, $n \gg l, k$

$$\frac{i}{\hbar}[p, x]_{n,n-l}\, e^{-i\omega_{n\leftarrow n-l}t} = \frac{\partial}{\partial J}\left(P_{n\leftarrow n-k}\, e^{-i\omega_{n\leftarrow n-k}t}\right)\frac{\partial}{\partial \theta}\left(X_{n-k,n-l}\, e^{-i\omega_{n-k,n-l}t}\right)$$
$$- \frac{\partial}{\partial J}\left(X_{n,n-(l-k)}\, e^{-i\omega_{n\leftarrow n-(l-k)}t}\right)\frac{\partial}{\partial \theta}\left(P_{n-(l-k),n-l}\, e^{-i\omega_{n-(l-k),n-l}t}\right)$$

we use the following identity: we suppress the time dependence

$$[P, X]_{n,n-l} = \sum_k \left(P_{n\,n-k} - P_{n-(l-k),\,n-l}\right) X_{n-k,n-l}$$
$$- \left(X_{n,n-(l-k)} - X_{n-k,n-l}\right) P_{n-(l-k),n-l}$$

$$= \hbar(l-k)\frac{\partial}{\partial J} P_{n,n-k}\; X_{n-k,n-l} - k\hbar\frac{\partial}{\partial J} X_{n,n-(l-k)}\, P_{n-(l-k),n-l}$$

Then acording $\;O_n(l)\, e^{il\omega_n t} \leftrightarrow O_{n-l,n}\, e^{-i\omega_{n-l \leftarrow n} t}$

$l>0$

$$O_n(-l) \leftrightarrow O_{n,n-l}\; e^{-i\omega_{n\leftarrow n-l}t}$$

$\cancel{\sum_k \frac{\partial}{\partial J} P_n(-k)}$

if $k>0$ $\qquad P_{n,n-k} \to P_n(-k)$

$k<0$ $\qquad P_{n,n-k} = P_{n,n+|k|} \longrightarrow P_{n-k}(-k) \sim P_n(-k)$

$X_{n-k,n-l} \to X_n(-l+k) \longrightarrow$ the difference between two indic

$\downarrow$

sum over the left and right indexs and then take the leading order

$$X_{n,n-(l-k)} \to X_n(-l+k)$$

$$P_{n-(l-k),n-l} \to P_n(-k)$$

$$\Rightarrow \sum_k \left[\frac{\partial}{\partial J} P_n(-k)\frac{\partial}{\partial \theta} X_n(-l+k) - \frac{\partial}{\partial J} X_n(-l+k)\frac{\partial}{\partial \theta} P_n(-k)\right] e^{il\omega_n}$$

$$= O_n(-l)\, e^{il\omega_n}$$

where $O = \frac{\partial}{\partial J}P\frac{\partial}{\partial \theta}X - \frac{\partial}{\partial J}X\frac{\partial}{\partial \theta}P = -\{P, X\}$

In principle, the above process applies to an arbitary f_1, f_2.

$$\Rightarrow \left(\frac{1}{i\hbar}[f_1, f_2]\right)_{n,n-l} \rightarrow \left(\{f_1, f_2\}\right)_n (-l)\, e^{il\omega_n t}$$

i.e $\quad \frac{1}{i\hbar}[f_1\ f_2] \leftarrow \{f_1, f_2\}$

This provides a systematic way for quantization

— Canonical quantization!