Lect 7. 1D harmonic oscilaters — coherent states!

In this lecture we will use two different methods to solve the eigen-wavefunction and states of 1D harmonic oscilaters.

§1.
$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2, \quad \text{define length unit } l = \sqrt{\frac{\hbar}{m\omega}}$$

$$\Rightarrow \quad H/\hbar\omega = -\frac{1}{2}\frac{d^2}{d(x/l)^2} + \frac{1}{2}(x/l)^2,$$

the eigen-equation $\quad H\psi = E_n \psi_n \quad \Rightarrow$

$$\left[-\frac{l^2}{2}\frac{d^2}{dx^2} + \frac{1}{2}\frac{x^2}{l^2}\right]\psi_n(x/l) = [E_n/\hbar\omega]\,\psi_n(x/l)$$

We first analyze the behavior of $\psi_n(x/l)$ at $x \to \pm\infty$. In this limit, we can neglect the constant $E_n/\hbar\omega$, we have

$$\left(-\frac{l^2}{2}\frac{d^2}{dx^2} + \frac{1}{2}\frac{x^2}{l^2}\right)\psi_n(x/l) \xrightarrow{x\to\pm\infty} 0$$

$$\Rightarrow \quad \psi_n(x/l) \sim e^{\pm\frac{1}{2}\frac{x^2}{l^2}} \quad \text{at the leading order}$$

we need the normalization condition that $\int_{-\infty}^{+\infty} dx \, |\psi_n(x/l)|^2$ finite, because all the states are bound states. we can only choose $\psi \sim e^{-\frac{1}{2}\frac{x^2}{l^2}}$.

Thus we try the solution

$$\psi_n = e^{-\frac{x^2}{2l^2}}\, u_n(x/l).$$

Plug this solution into the Schrödinger Eq, we have

$$\frac{d}{d(x/\ell)}\left[e^{-\frac{1}{2}(x/\ell)^2} u_n(x/\ell)\right] = -\frac{x}{\ell} e^{-\frac{1}{2}\frac{x^2}{\ell^2}} u_n(x/\ell) + e^{-\frac{x^2}{2\ell^2}} \frac{d}{d(x/\ell)} u_n(x/\ell)$$

$$\frac{d^2}{d(x/\ell)^2}\left[e^{-\frac{1}{2}(x/\ell)^2} u_n(x/\ell)\right] = (x/\ell)^2 e^{-\frac{1}{2}\frac{x^2}{\ell^2}} u_n(x/\ell) + e^{-\frac{x^2}{2\ell^2}} \frac{d^2}{d(x/\ell)^2} u_n$$

$$- 2(x/\ell) e^{-\frac{x^2}{2\ell^2}} \frac{d}{d(x/\ell)} u_n - e^{-\frac{x^2}{2\ell^2}} u_n$$

$$\Rightarrow \quad \frac{d^2}{d(x/\ell)^2} u_n - 2(x/\ell)\frac{d}{d(x/\ell)} u_n - \left(\frac{2E_n}{\hbar\omega} - 1\right) u_n = 0$$

define $z = x/\ell$, $\lambda_n = 2E_n/\hbar\omega$ $\Rightarrow$

$$\boxed{\frac{d^2}{dz^2} u_n(z) - 2z\frac{d}{dz} u_n(z) - (\lambda_n - 1)\, u_n(z) = 0}$$

*) Some results quoted from the study of Hermite polynomials.

only at $\lambda_n - 1 = 2n$, with "n" is a non-negative integers, we have polynomial solutions $H_n(z)$. The generation function for Hermite polynomials is

$$e^{-s^2 + 2zs} = \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} s^n$$

$$\Rightarrow \quad H_n(z) = \frac{d^n}{ds^n} e^{-s^2+2zs}\Big|_{s=0} = e^{z^2}\frac{d^n}{ds^n} e^{-(s-z)^2}\Big|_{s=0} = (-)^n e^{z^2}\frac{d^n}{dz^n} e^{-(s-z)^2}\Big|_{s=0}$$

$$\boxed{H_n(z) = (-)^n e^{z^2}\frac{d^n}{dz^n} e^{-z^2}}$$

a few examples $H_n(z)$:

$$H_0(z) = 1, \quad H_1(z) = 2z, \quad H_2(z) = 4z^2 - 2, \quad H_3(z) = 8z^3 - 12z.$$

They satisfy the relation:

$$\begin{cases} H_{n+1} - 2z H_n + H_{n-1} = 0 \\ \frac{d}{dz} H_n = 2n H_{n-1} \end{cases}$$

they are normalized

$$\int_{-\infty}^{+\infty} H_m(z) H_n(z) e^{-z^2} dz = \sqrt{\pi}\, 2^n n!\, \delta_{mn}$$

$\Rightarrow$ normalized solution for $E_n = (n + 1/2)\hbar\omega$,

$$\boxed{\psi_n(x) = \left[\frac{1}{\sqrt{\pi}\, 2^n n!\, \ell}\right]^{1/2} H_n(x/\ell)\, e^{-\frac{x^2}{2\ell^2}},}$$

:

$$\psi_0(x) = \frac{1}{\pi^{1/4} \ell^{1/2}} e^{-\frac{x^2}{2\ell^2}} \qquad \text{even}$$

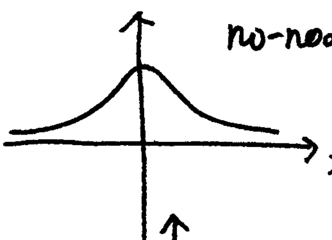
$$\psi_1(x) = \frac{\sqrt{2}}{\pi^{1/4} \ell^{1/2}} \frac{x}{\ell} e^{-\frac{x^2}{2\ell^2}} \qquad \text{odd}$$

$$\psi_2(x) = \frac{1}{\pi^{1/4}} \sqrt{\frac{1}{2\ell}} \left[2\left(\frac{x}{\ell}\right)^2 - 1\right] e^{-\frac{x^2}{2\ell^2}} \qquad \text{even}$$

$$\psi_3(x) = \frac{1}{\pi^{1/4}} \sqrt{\frac{3}{\ell}} \left[\frac{2}{3}\left(\frac{x}{\ell}\right)^2 - 1\right]\left(\frac{x}{\ell}\right) e^{-\frac{x^2}{2\ell^2}} \qquad \text{odd}$$

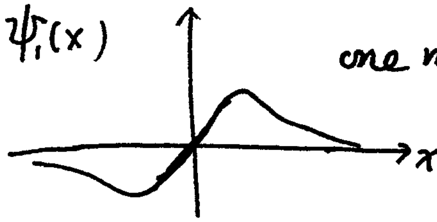
Parity $\quad \psi_n(-x) = (-)^n \psi_n(x)$:

① $\psi_0(x)$



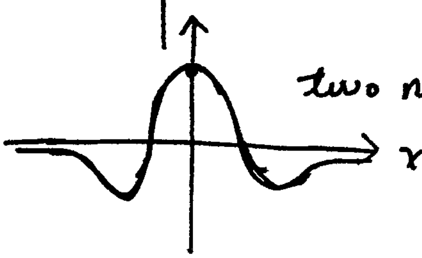
no-node

② $\psi_1(x)$



one node

③ $\psi_2(x)$

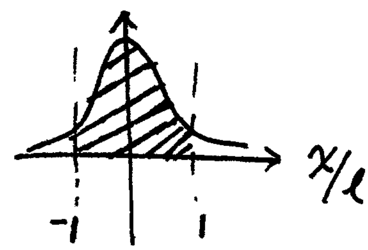


two nodes

......

Look at the ground state: Gaussian pocket.

- The classic regin is at $|x/\ell| \leq 1$, the propability that the particle lying outside the classic regin



$$\int_1^\infty e^{-z^2} dz \Big/ \int_0^\infty e^{-z^2} dz \simeq 16\% .$$

Algebraic Solution

define $a = \frac{1}{\sqrt{2}}\left(x/\ell + i\frac{p\ell}{\hbar}\right)$, and $a^+ = \frac{1}{\sqrt{2}}\left[x/\ell - i\frac{p\ell}{\hbar}\right]$.

easy to check $[a, a^+] = 1$.

Ex: Please check $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 = \frac{\hbar\omega}{2}[aa^+ + a^+a] = \hbar\omega[a^+a + 1/2]$

- , a^+a is called the phonon number operator.

Please note: we have changed our viewpoint. Before, we viewed

$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$ as a single particle problem with many eigenmodes, and the particle is not in the free space. But now, we rewrite $H = \hbar\omega[a^+a + 1/2]$ it becomes a single-mode phonon problem, and the number of phonons (n) is related to the n-th excited state.

Now let us solve the spectra of a^+a. First, a^+a is non-negative.

- ① For any state $|\psi\rangle$, $\langle\psi|a^+a|\psi\rangle = |a|\psi\rangle|^2 \geq 0$, thus all the eigenvalues of a^+a should be non-negative.

2 Suppose $a^\dagger a \, |n\rangle = n|n\rangle$, where n is eigenvalue with $n \geq 0$.

please check $[a^\dagger a, a^\dagger] = a^\dagger$, $[a^\dagger a, a] = -a$

$$\Rightarrow \qquad a^\dagger a \, (a|n\rangle) = (n-1)\,(a|n\rangle),$$

thus $a|n\rangle$ is also $a^\dagger a$'s eigenstate, with the eigenvalue $n-1$.

~~thus~~ we have a series of eigenstates

$|n\rangle$, $a|n\rangle$, $a^2|n\rangle$, ..., with eigenvalues $n, n-1, n-2, \cdots$.

Thus n has to be an integer number, such that this sequence has to be terminated at $n=0$. we have $|0\rangle$, but $a|0\rangle = 0$.

Now we ~~start~~ from $|0\rangle$, and apply $a^\dagger$ successively, then we arrive at the sequence

$|0\rangle$, $a^\dagger|0\rangle$, $(a^\dagger)^2|0\rangle$, $\cdots$, We define $|n\rangle = N_n (a^\dagger)^n |0\rangle$,

where N_n is the normalization factor, such that $\langle n|n\rangle = 1$.

$$\boxed{\begin{aligned} (a^\dagger a)\, a|n\rangle &= (n-1)\, a|n\rangle \\ (a^\dagger a)\, a^\dagger|n\rangle &= (n+1)\, a^\dagger|n\rangle \end{aligned}}$$

$\leftarrow$ please prove.

now, we determine N_n.

$$\langle n|n\rangle = \left|\frac{N_n}{N_{n-1}}\right|^2 \langle n-1|\, a\, a^\dagger\, |n-1\rangle = \left|\frac{N_n}{N_{n-1}}\right|^2 \langle n-1|(a^\dagger a+1)|n-1\rangle$$

$$= n\left|\frac{N_n}{N_{n-1}}\right|^2 = 1.$$

we can choose N_n to be real $\Rightarrow$ $N_n = \sqrt{n}\, N_{n-1}$.

with the definition $\Rightarrow N_0 = 1. \Rightarrow N_n = \sqrt{n!}$

$$\Rightarrow \boxed{\begin{aligned} |n\rangle &= \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle \\ a^\dagger |n\rangle = \sqrt{n+1}\ |n+1\rangle, &\quad a|n\rangle = \sqrt{n-1}\ |n-1\rangle. \end{aligned}}$$

Ex: 1) prove the matrix elements

$$\langle m | x | n \rangle = \frac{\ell}{\sqrt{2}} \left(\sqrt{n+1}\, \delta_{m,n+1} + \sqrt{n}\, \delta_{m,n-1} \right)$$

$$\langle m | P | n \rangle = \frac{i\hbar}{\sqrt{2}\,\ell} \left(\sqrt{n+1}\, \delta_{m,n-1} - \sqrt{n}\, \delta_{m,n-1} \right)$$

2) Check $[x, p] = i\hbar$, using the above matrix elements

3) prove that for the state $|n\rangle$,

$$\langle n | x^2 | n \rangle = \ell^2 (n + 1/2) \quad \langle n | p^2 | n \rangle = \frac{\hbar^2}{\ell^2} (n + 1/2)$$

$$\Rightarrow \Delta X \cdot \Delta P = (n + 1/2)\, \hbar$$

at $n = 0$, the uncertain relation reaches the minimum.

*) Now we need to work out the wavefunction.

For ground states $|0\rangle$, we have $a|0\rangle = 0$.

$$\langle x | a | 0 \rangle = \langle x | \frac{\hat{x}}{\ell} + i\ell \hat{p} | 0 \rangle = \left[\frac{x}{\ell} + \frac{\partial}{\partial x} \cdot \ell \right] \langle x | 0 \rangle = 0$$

$$\Rightarrow \boxed{\psi_0(x) = \langle x | 0 \rangle = \frac{1}{\pi^{1/4} \ell^{1/2}}\, e^{-\frac{x^2}{2\ell^2}}}.$$

and

$$\psi_n(x) = \frac{1}{\pi^{1/4} \ell^{1/2}} \frac{1}{\sqrt{n!}} \left(\frac{x}{\ell} - \ell \frac{d}{dx} \right)^n e^{-\frac{x^2}{2\ell^2}}$$

*) Cohrent states

We first prove Baker-Hausdorff lemma:

$$e^{i\lambda G} A e^{-i\lambda G} = A + i\lambda [G,A] + \frac{(i\lambda)^2}{2!}[G,[G,A]] + \cdots + \left(\frac{i^n\lambda^n}{n!}\right)[G,[G,\cdots[G,A]]\cdots] + \cdots$$

Proof: define $O(\lambda) = e^{i\lambda G} A e^{-i\lambda G}$, $\Rightarrow \frac{d}{d\lambda} O = i e^{i\lambda G}[G,A] e^{-i\lambda G}$

$$= i[G, O(\lambda)]$$

$\Rightarrow O(\lambda) = O(0) + i\int_0^\lambda d\lambda_1 [G, O(\lambda_1)]$, and $O(0) = A$

$$\Rightarrow O(\lambda) = A + i\int_0^\lambda d\lambda_1 [G,A] + i^2\int_0^\lambda d\lambda_1 \int_0^{\lambda_1} d\lambda_2 [G,[G,O(\lambda_2)]]$$

$$= A + i\int_0^\lambda d\lambda_1 [G,A] + \cdots \quad i^n \int_0^\lambda d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \cdots \int_0^{\lambda_{n-1}} d\lambda_n [G[G,\cdots[G,A]]\cdot + \cdots$$

plug in $\int_0^\lambda d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \cdots \int_0^{\lambda_{n-1}} d\lambda_n = \frac{\lambda^n}{n!}$, we arrive at the above lenna.

or
$$\boxed{e^B A e^{-B} = A + [B,A] + \frac{1}{2!}[B,[B,A]] + \cdots + \frac{1}{n!}[B,[B,\cdots[B,A]]\cdot] + \cdots}$$

Define coherent states as eigenstates of a, i.e. $a|\alpha\rangle = \alpha|\alpha\rangle$
where α can be a complex number.

Using Baker-Hausdorff Lenna, $e^{-\alpha a^\dagger} a \, e^{\alpha a^\dagger} = a - [\alpha a^\dagger, a] = a + \alpha$

$$\Rightarrow e^{-\alpha a^\dagger} a \, e^{\alpha a^\dagger}|0\rangle = \alpha|0\rangle \Rightarrow a\left(e^{\alpha a^\dagger}|0\rangle\right) = \alpha\left(e^{\alpha a^\dagger}|0\rangle\right)$$

$$\Rightarrow |\alpha\rangle = N_\alpha \, e^{\alpha a^\dagger}|0\rangle$$

$$\langle\alpha|\alpha\rangle = |N_\alpha|^2 \langle 0| e^{\alpha^* a} e^{\alpha a^\dagger} |0\rangle$$

using Baker - Hausdorff: we prove $e^{\alpha^* a} e^{\alpha a^\dagger} e^{-\alpha^* a} = ?$

$$e^B e^A e^{-B} = e^A + [B, e^A] + \cdots + \frac{1}{n!}[B, \cdots [B, e^A]] + \cdots$$

now $B = \alpha^* a$, $e^A = e^{\alpha a^\dagger}$ $\Rightarrow$ $[B, e^A] = \alpha^* [a, e^{\alpha a^\dagger}]$

according to $e^{-\alpha a^\dagger} a\, e^{\alpha a^\dagger} = a + \alpha$ $\Rightarrow$ $a e^{\alpha a^\dagger} = a e^{\alpha a^\dagger} + \alpha e^{\alpha a^\dagger}$

$$\Rightarrow [B, e^A] = \alpha^*[a, e^{\alpha a^\dagger}] = \alpha^* \alpha\, e^{\alpha a^\dagger} = \alpha^*\alpha\, e^A$$

$$\Rightarrow e^B e^A e^{-B} = e^A \sum_{n=0}^{\infty} \frac{(\alpha^*\alpha)^n}{n!} = e^A e^{|\alpha|^2} \Rightarrow e^{\alpha^* a} e^{\alpha a^\dagger} = e^{\alpha a^\dagger} e^{\alpha^* a} e^{|\alpha|^2}$$

$$\Rightarrow \langle\alpha|\alpha\rangle = |N_\alpha|^2 e^{|\alpha|^2} \langle 0| e^{\alpha a^\dagger} e^{\alpha^* a} |0\rangle = |N_\alpha|^2 e^{|\alpha|^2} = 1$$

$\Rightarrow$ normalization factor $N_\alpha = e^{-|\alpha|^2/2}$

$$\Rightarrow \boxed{|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha)^n}{\sqrt{n!}} |n\rangle}$$

$$\boxed{\begin{aligned} e^A e^B &= e^B e^A e^{[A,B]} \\ &= e^{A+B} e^{\frac{1}{2}[A,B]} \end{aligned}}$$

if $[A, B]$ commutes with A, and B.

Ex: Please use the fact of $a|\alpha\rangle = \alpha|\alpha\rangle$, prove that for state $|\alpha\rangle$, define $\overline{\Delta X^2} = \langle\alpha|X^2|\alpha\rangle - (\langle\alpha|x|\alpha\rangle)^2$

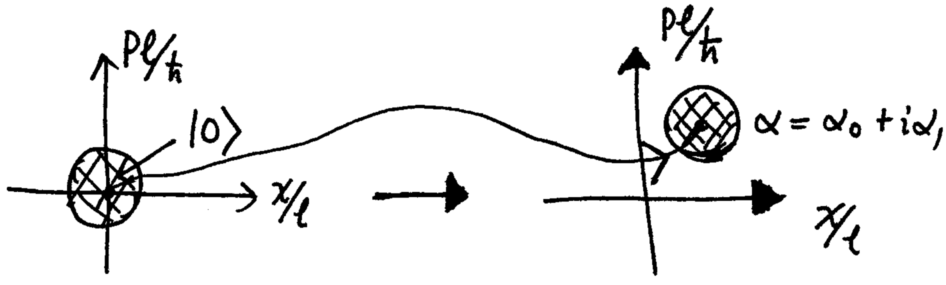
and $\overline{\Delta p^2} = \langle\alpha|p^2|\alpha\rangle - (\langle\alpha|p|\alpha\rangle)^2$

we have $\sqrt{\overline{\Delta X^2}}\sqrt{\overline{(\Delta p)^2}} = \frac{\hbar}{2}$.

What's the physical meaning of $|\alpha\rangle$, define $\alpha = \alpha_0 + i\alpha_1$

$$\Rightarrow \left[\frac{(x - l\alpha_0)}{l} + l\left(\frac{\partial}{\partial x} - i\frac{\alpha_1}{l}\right)\right] \psi_\alpha(x) = 0$$

$$\Rightarrow \psi_\alpha(x) = \frac{1}{\pi^{1/4} l^{1/2}} e^{-\frac{(x - l\alpha_0)^2}{2l^2} + i\frac{\alpha_1}{l}x}$$



*) equation of motion in the Heinsenberg picture.

$$a^\dagger(t) = e^{iHt} a^\dagger e^{-iHt} \qquad a(t) = e^{iHt} a e^{-iHt}$$

$$= e^{i a^\dagger a t\omega} a^\dagger e^{-i a^\dagger a t\omega} \qquad = e^{i a^\dagger a t\omega} a e^{-i a^\dagger a t\omega}$$

$$\Rightarrow \begin{cases} \frac{d}{dt} a^\dagger(t) = i\omega a^\dagger & a^\dagger(t) = a^\dagger e^{i\omega t} \\ \frac{d}{dt} a(t) = -i\omega a & a(t) = a e^{-i\omega t} \end{cases}$$

$$\Rightarrow x(t) = \frac{l}{\sqrt{2}}\left[a^\dagger(t) + a(t)\right] = \frac{l}{\sqrt{2}}\left[a^\dagger(0) e^{i\omega t} + a(0) e^{-i\omega t}\right]$$

$$= x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t$$

$$p(t) = \frac{\hbar}{\sqrt{2} i}\left[a^\dagger(t) - a(t)\right] = -m\omega x(0)\sin\omega t + p(0)\cos\omega t \;!$$

Extensions:

(X) $\mathcal{U}(1)$ or $SO(2)$ symmetry in phase space

Harmonic oscillator Hamiltonian is invariant, under the transformation

$$\begin{cases} a^\dagger \to a^\dagger e^{i\theta} \\ a \to a e^{-i\theta} \end{cases} \quad \text{or} \quad \begin{pmatrix} x'/\ell \\ \frac{\ell p'}{\hbar} \end{pmatrix} = \begin{Bmatrix} \frac{x}{\ell}\cos\theta + \frac{\ell p}{\hbar}\sin\theta \\ -\frac{x}{\ell}\sin\theta + \frac{\ell p}{\hbar}\cos\theta \end{Bmatrix}.$$

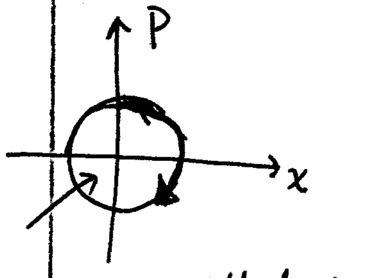
This transformation can be generated by $\boxed{R = e^{-i\theta a^\dagger a}}$

i.e. $R^\dagger(\theta)\, a^\dagger R = e^{i\theta a^\dagger a}\, a^\dagger\, e^{-i\theta a^\dagger a} = e^{i\theta} a^\dagger$

The generator of this $\mathcal{U}(1)$ symmetry is nothing but the Hamitonian!
proportional

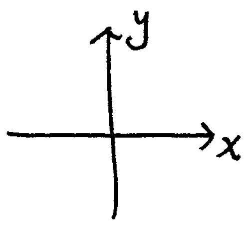
This is an angular momentum in phase-space: (but $x-p$ conjugate)

$$L_{\text{phase-space}} = x\,\pi_p - p\,\pi_x, \quad \text{according to} \quad \pi_x = p$$
$$\pi_p = -x.$$
$$= -(x^2+p^2) \quad \text{which is non-negative!}$$



$\longrightarrow$ Compare with the case of usual 2D

harmonic oscillator's motion in phase space is Chiral!!



$$\Rightarrow L_z = xP_y - yP_x$$
$$= -i\hbar \frac{\partial}{\partial\varphi}$$

L_z's spectra $-\infty, \cdots -1, 0, 1, \cdots +\infty$,

take all the integer values.

(no-chiral)

what's the symmetry of the 2D, or more generally nD harmonic oscilator?

$$H = \hbar\omega\left[\frac{n}{2} + (a_1^\dagger, \cdots a_n^\dagger)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\right]$$

(quadratic hamitonican can be variable-separated!)

introducing Unitary transformation $\begin{pmatrix} a_1' \\ \vdots \\ a_n' \end{pmatrix} = U\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ ⇒ The harmitonian is invariant,

i.e. the sysmetry is $U(n)$, not just $SO(n)$. (You can take $n=2$ or 3 as examples).

The degeneracy pattern for n Dimmensional harmonic oscillator,

$n_1 + n_2 + \cdots n_n = m$, where n_i is is the quantum number along i-th direction,

n_i is non-negative integer.

→ number of degeneracy

$$g_n(m) = \binom{m+n-1}{n-1} = \frac{(m+n-1)!}{m!\,(n-1)!}$$

0|0|00

m - balls

$n-1$ - baffles

① 1D : $g_1(m) = 1$

② 2D : $g_2(m) = m+1$

③ 3D : $g_3(m) = \frac{(m+2)!}{m!\,2!} = \frac{(m+2)(m+1)}{2}$

more formally, the m-th level of nD harmonic oscillator belongs to the $\underbrace{\square\square\square\square\square}_{m}$ representation of $SU(n)$ group.

$(m = 0, 1, 2, \cdots)$.

Q: what are the generators of the $U(n)$ transformation? Let's only take the 2D case as an example.

The rotation in real space $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, $\begin{pmatrix} P_x' \\ P_y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} P_x \\ P_y \end{pmatrix}$

Such a rotation is generated by the usual angular momentum L_{xy},

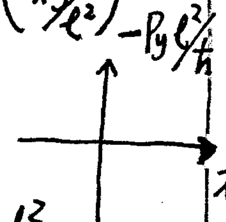
the rotation in $\begin{pmatrix} x-y \\ P_x-P_y \end{pmatrix}$ planes, i.e. $L_{xy} = \dfrac{xP_y - yP_x}{\hbar} = -i(a_x^\dagger a_y - a_y^\dagger a_x)$

The rotation operator is $R_{xy}(\theta) = e^{-iL_{xy}\theta}$.

For harmonic potential $\frac{P_y^2}{2m} + \frac{1}{2}m\omega^2 y^2$, we can define a canonical transformation $\left(y/l,\ lP_y/\hbar\right) \rightarrow \left(-\frac{lP_y}{\hbar},\ y/l\right)$.

or we define rotation in the $x \leftrightarrow \frac{-P_y l^2}{\hbar}$; $P_x - \hbar y/l^2$ planes

as $\begin{pmatrix} x' \\ -\frac{P_y' l^2}{\hbar} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ -\frac{P_y l^2}{\hbar} \end{pmatrix}$ and $\begin{pmatrix} P_x' \\ \frac{\hbar y'}{l} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} P_x \\ \hbar y/l^2 \end{pmatrix}$



This transformation is generated in the plane: $x - -P_y l^2/\hbar$,

$$Q_{xy} = \frac{1}{\hbar}\left[x \cdot \frac{\hbar y}{l^2} - \left(\frac{-P_y l^2}{\hbar}\right) P_x\right] = \frac{xy}{l^2} + \frac{P_x P_y l^2}{\hbar^2}$$

$$= (a_x^\dagger a_y + a_y^\dagger a_x)$$

Similarly, we can have the rotations in $(x\ \ P_x l^2/\hbar)$ and $(y.\ P_y l^2/\hbar)$ planes. $Q_{xx} = a_x^\dagger a_x$, $Q_{yy} = a_y^\dagger a_y$

we decompose $U(2) = U(1) \otimes SU(2)$

$U(1) \rightarrow a_x^\dagger a_x + a_y^\dagger a_y$

$SU(2) \rightarrow \begin{cases} \frac{1}{2}(a_x^\dagger a_x - a_y^\dagger a_y) \\ \frac{1}{2}(a_x^\dagger a_y \pm a_y^\dagger a_x) \end{cases}$