Quantum Mechanics 2025 HW1

Due 09/09 in Class

September 4, 2025

Problem 1. Canonical Transformation

In class we have shown one particular type of canonical transformation preserving the form of Hamilton equations under the transformation from (q, p) to (Q, P). It is generated by $F(q, Q, t)$ as

$$dF = pdq - PdQ + (H' - H)dt, \tag{1}$$

which gives rise to $p = \frac{\partial F}{\partial q}$, $P = -\frac{\partial F}{\partial Q}$, and $H' = H + \frac{\partial F}{\partial t}$.

1) Please find another type of generation function $\Phi(q, P, t)$. Express p and Q, and $H' - H$ in terms of the partial derivatives of $\Phi(q, P, t)$.

2) A convenient choice of canonical transformation is via the time-evolution of (q, p), *i.e.*,

$$q_{t+\tau} = q(q_t, p_t, t), \quad p_{t+\tau} = p(q_t, p_t, t), \tag{2}$$

where (q_t, p_t) represents the location of the system in the phase space at time t, and $(q_{t+\tau}, p_{t+\tau})$ is that at time $t + \tau$. The time evolution is govern under the Hamiltonian $H(q, p, t)$.

Check that the above canonical transformation can be generated via $\Psi = -S(q_{t+\tau}, q_t, \tau) + p_{t+\tau}q_{t+\tau}$, where S is the action along the physical path,

$$S(q_{t+\tau}, q_t, \tau) = \int_t^{t+\tau} L(q, \dot{q}, t)dt. \tag{3}$$

3) Define the Jacobian D as

$$D = \begin{vmatrix} \frac{\partial q_{t+\tau}}{\partial q_t} & \frac{\partial q_{t+\tau}}{\partial p_t} \\ \frac{\partial p_{t+\tau}}{\partial q_t} & \frac{\partial p_{t+\tau}}{\partial p_t}. \end{vmatrix} \tag{5}$$

Prove that $D = 1$.

Problem 2. Hamilton-Jacobi equation

Define $S(q,t)$ as the action from (q_0, t_0) to (q,t) along the physical path. The system Hamiltonian is $H(q,p)$.

1) Prove that

$$\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}) = 0 \tag{6}$$

Suppose the solution can be represented as

$$S = S(t, q, \alpha) + C \tag{7}$$

where α is a constant for solving S and C is an arbitrary additive constant.

Treat S as the generation function, and perform the canonical transform (q,p) to (β, α) where β is the new coordinate and α is the new momentum.

2) Please find the expression of β and prove $H' = 0$ in this representation. Hence $\dot{\alpha} = 0$ and $\dot{\beta} = 0$, such that we identify a new conserved quantity β.

Problem 3. Harmonic Oscillator

Consider a 1D harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2. \tag{8}$$

1) Please show that its action as a function of the final position q and the final time t can be expressed as

$$S(x,t;E) = \int^x dxp - Et + const, \tag{9}$$

where $E = \frac{1}{2}m\omega^2 A^2$ with A the amplitude, and $p = \pm m\omega\sqrt{A^2 - x^2}$. The $\pm$ is determined by the direction of motion.

2) Calculate $\frac{\partial S}{\partial t_f}$ and $\frac{\partial S}{\partial x_f}$.

3) Calculate $\beta = \frac{\partial S}{\partial E}$. Prove β is a constant. Explain the physical meaning of β.

Problem 4. Planck's derivation of black body radiation

Read my lecture note and follow Planck's idea: By checking $u(\omega)$'s behavior at high ω and low ω-limits, derive

$$u(\omega) = \frac{\hbar\omega}{e^{\hbar\omega/k_B T} - 1}, \tag{10}$$

where $u(\omega)$ is the average energy for an EM mode at the frequency ω.

Problem 5. Heisenberg's magic

Read my lecture note. Prove the existence of the zero point motion of a harmonic oscillator

$$(\dot{x})^2_{nn} = (n + 1/2)\frac{\hbar\omega}{m}, \tag{11}$$

where n takes integer values starting from 0.