Quantum Mechanics 2025 HW3

Due 09/30 in Class

September 23, 2025

Problem 1. A quantum particle in an infinitely deep potential well

Consider a quantum particle with the mass m moving in a quantum well with the potential of

$$V(x) = \begin{cases} +\infty, |x| \geq L/2, \\ 0, |x| < L/2. \end{cases} \tag{1}$$

1) Solve the eigen-energies $E_n^{\pm}$ and the associated normalized eigen-wavefunctions $\psi_n^{\pm}(x)$. $\pm$ are the parity number; $E_1^+ < E_2^+ < E_3^+ < ...$ and $E_1^- < E_2^- < E_3^- < ....$

2) Consider an initial state

$$\psi(t=0) = \frac{1}{\sqrt{2}} \left(\psi_1^+(x) + \psi_1^-(x) \right). \tag{2}$$

Please calculate its time-evolution $\psi(t)$.

3) Calculate $\sqrt{(\Delta x)^2}$ and $\sqrt{(\Delta p)^2}$ in each state. Check whether they satisfy $\sqrt{(\Delta x)^2}\sqrt{(\Delta p)^2} \geq \hbar/2$.

Problem 2. δ-function

1) Prove that $\lim_{a\to 0} = \frac{1}{\pi}\frac{a}{x^2+a^2} = \lim_{a\to 0}\frac{1}{a\sqrt{\pi}}e^{-x^2/a^2} = \delta(x)$.

2) Prove that $\frac{1}{x\pm i\epsilon} = P(1/x) \mp i\pi\delta(x)$, where P means the principal value of integration and $\epsilon \to 0^+$.

3) Please figure out what is $\delta(x^2-a^2)$. In general what is $\delta(f(x))$?

4) Please calculate $\int_{-\infty}^{+\infty}\frac{d}{dx}\delta(x-a)\,(xf(x)) =$? Similarly, $\int_{-\infty}^{+\infty}\frac{d^2}{d^2x}\delta(x-a)\,(x^2f(x)) =$?

Problem 3. Momentum representation

In class, we explained wave mechanics is in fact the coordinate representation of quantum mechanics by identifying $\psi(x) = \langle x|\psi\rangle$. We also proved in class that $\langle x|\hat{p}|\psi\rangle = -i\hbar\lim_{\Delta x\to 0}\frac{\langle x+\Delta x|\psi\rangle - \langle x|\psi\rangle}{\Delta x} = -i\hbar\frac{d}{dx}\psi(x)$.

1) Please derive that $\langle x|\hat{p}^2|\psi\rangle = -\hbar^2\frac{d^2}{dx^2}\psi(x)$.

2) Prove that the Hamiltonian of an harmonic oscillator in the coordinate representation $\langle x|\hat{H}|\psi\rangle = (-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2)\psi(x)$.

3) Define $\psi(p) = \langle p|\psi\rangle$. Then please derive that $\langle p|\hat{x}|\psi\rangle = i\hbar\frac{d}{dp}\psi(p)$.

4) Please derive the same Hamiltonian of the harmonic oscillator in the momentum representation $\langle p|H|\psi\rangle =$?. Please represent the right-hand side as operators acting on $\psi(p)$.

Problem 4. Orbital angular momentum

The operators of orbital angular momentum L_i with $i = x, y, z$ are defined as

$$L_i = \epsilon_{ijk} x_j p_k. \tag{3}$$

1) Prove that $[L_i, L_j] = i\epsilon\hbar L_k$. Define $L^2 = L_x^2 + L_y^2 + L_z^2$. Prove that $[L^2, L_i] = 0$.

2) Prove that $[L_i, x_j] = i\epsilon_{ijk}\hbar x_k$ and $[L_i, p_j] = i\epsilon_{ijk}\hbar p_k$.

3) Prove that $[L_i, r^2] = 0$ where $r^2 = x^2 + y^2 + z^2$ and $[L_i, p^2] = 0$.

4) Write down the expressions of L_i in coordinate representation, and also those in momentum representation.

Problem 5. Complete set of Mechanical variables

For a physical problem, we often find a maximal set of observables which commute each other. Then they share the common eigenstates. For a hydrogen atom,

$$H = -\tfrac{\hbar^2}{2m}\nabla^2 - \tfrac{e^2}{r}. \tag{4}$$

This set include Hamiltonian H itself, the angular momentum square L^2, the z-component of angular momentum, *i.e.*, (H, L^2, L_z).

Prove that $[H, L_i] = 0$ for $i = x, y, z$, and $[H, L^2] = 0$. Later we will use (n, l, m) to denote eigenvalues of E_n, $l(l+1)\hbar^2$, $m\hbar$ for (H, L^2, L_z), respectively.

Problem 6. Gaussian and uncertainty principle

Consider the wavefunction that $\psi(x) = \langle x|\psi\rangle = Ae^{-\frac{x^2}{2l^2}}$. Please derive the normalization factor A according to $\int_{-\infty}^{+\infty} dx|\psi(x)|^2 = 1$.

1) Calculate $\sqrt{(\Delta x)^2}$ and $\sqrt{(\Delta p)^2}$. Please verify they satisfy the lower bond of the uncertainty principle.

2) We define $\hat{p}|p\rangle = p|p\rangle$, and set its normalization condition that $\int_{-\infty}^{+\infty} dx|\langle x|p\rangle|^2 = 1$. Please find the expression of $\psi(p) = \langle p|\psi\rangle$. Then calculate $\sqrt{(\Delta p)^2}$ and $\sqrt{(\Delta x)^2}$ in the momentum representation again.