# Quantum Mechanics 2025 HW4

Due 10/09 in Class

September 24, 2025

**Problem 1. Probability current**

Consider the Schrödinger equaiton

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r},t)\right)\psi(\mathbf{r},t). \tag{1}$$

Define the probability density $\rho(x,t) = \psi^\dagger(x,t)\psi(x,t)$. In order to have the probability conservation, i.e., $\frac{\partial}{\partial t}\rho + \nabla\cdot\mathbf{j} = 0$, how should we define the probability current $\mathbf{j}(\mathbf{r},t)$?

**Problem 2. time-evolution**

Consider a physical observable $O$, and its expectation value $\bar{O}(t) = \langle\psi(t)|O|\psi(t)\rangle$. The Hamiltonian of the system is time-independent.

Prove the following statements:

1) If $[O,H] = 0$, then $\frac{d}{dt}\bar{O}(t) = 0$.

2) If $|\psi(t)\rangle$ itself is the time-evolution of an eigen-state of $H$, then $\frac{d}{dt}\bar{O}(t) = 0$.

**Problem 3. The $f$-sum rule**

Consider a particle in 1D whose Hamiltonian is given by

$$H = p^2/2m + V(x). \tag{2}$$

By calculating $[[H,x],x]$ prove

$$\sum_l |\langle l|x|n\rangle|^2(E_l - E_n) = \hbar^2/2m, \tag{3}$$

where $|n\rangle$ and $|k\rangle$ are energy eigenstates of $H$ with the eigenvalues $E_n$ and $E_k$, respectively. This is essentially the quantization condition that Heisenberg used to establish quantum mechanics.

**Problem 4. The double $\delta$-potential**

A particle of mass $m$ moves in 1D $x$ under the potential $V(x)$.

$$V(x) = \gamma \left( \delta(x + \tfrac{a}{2}) + \delta(x - \tfrac{a}{2}) \right). \tag{4}$$

1) Consider the scattering states: At $x < -\frac{a}{2}$, it is a superposition of an incident wave superposed with a reflection wave $\psi(x) = e^{ikx} + Re^{-ikx}$. At $x > \frac{a}{2}$, it is the transmission wave $\psi(x) = Se^{ikx}$. Find the expression of the scattering amplitude $R$ and $S$.

2) Please find the conditions for the perfect transmission, i.e., $|S| = 1$.

3) Please find the bound state solutions at $\gamma < 0$ Express $R$ and $S$ in terms of $E$. Please show that the bound state information shows up as the pole of the scattering amplitude.

**Problem 5. Bound states**

Consider a 1D potential $V(x) \leq 0$ for $+\infty < x < +\infty$. As $|x| \to +\infty$, $V(x) \to 0$, and $V(x = 0) \neq 0$. Please prove that for the Hamiltonian

$$H = -\tfrac{\hbar^2}{2m} \tfrac{d^2}{dx^2} + V(x), \tag{5}$$

there must exist at least one bound state $\psi_b(x)$ whose eigen-energy $E_b < 0$.

(Hint: Consider the following problem. Suppose two potential functions $V_{1,2}(x)$ satisfying $V_1(x) < V_2(x)$ for $+\infty < x < +\infty$, then prove that the ground state energy $E_{g1}$ for the system with potential $V_1$ is smaller than that of $E_{g2}$ for the system with potential $V_2$.)