Quantum Mechanics 2025 HW8

Due 11/18 in Class

November 11, 2025

Problem 1. Heisenberg picture

1) Please prove Eq. (1)

$$\begin{aligned} x^H(t) =& \ x^S \cos\omega t + \frac{p^S}{m\omega}\sin\omega t, \\ p^H(t) =& \ -m\omega x^S \sin\omega t + p^S \cos\omega t. \end{aligned} \tag{1}$$

for harmonic oscillators by directly using

$$x^H(t) = e^{\frac{\mathrm{i}}{\hbar}H^S t} x^S e^{-\frac{\mathrm{i}}{\hbar}H^S t}, p^H(t) = e^{\frac{\mathrm{i}}{\hbar}H^S t} p^S e^{-\frac{\mathrm{i}}{\hbar}H^S t}. \tag{2}$$

(Hint: You may use the Baker–Hausdorff lemma on page 95 in Sakurai and Napolitano's book.)

Problem 2. Interaction picture

Study **Interaction Picture** on your own and write down all the derivations from the lecture notes(**Lecture 6: Pictures, QuanLect14_Symmetries**).

Problem 3. Rotation Operator

Please refer to Section 2 **Rotation Operator** of the lecture notes **QuanLect14_Symmetries** for background, and complete the following problems.

1) Prove that $\alpha = \hbar$.

2) Prove $[S_i, S_j] = \mathrm{i}\epsilon_{ijk}\hbar S_k$.

3) From $D^\dagger(g) L_i D(g) = g_{ij} L_j$,please derive that $[L_i, L_j] = \mathrm{i}\epsilon_{ijk}\hbar L_k$,which is consistent with the direct calculation using the canonical quantization condition.

4) From $D^\dagger(g) p_i D(g) = g_{ij} p_j$,please derive that $[L_i, p_j] = \mathrm{i}\epsilon_{ijk}\hbar p_j$.

Problem 4. Pauli Matrices

Based on Section 3 **Pauli matrices for spin-$\frac{1}{2}$** particles in the lecture notes **QuanLect14_Symmetries**, solve the following problems.

1) Prove the anti-commutation relation $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$ which is independent of the concrete representation.

2)Prove that for the rotation operator from the spin part $D_s(n,\theta) = e^{-\frac{\mathrm{i}}{2}\theta\vec{\sigma}\cdot\vec{n}}$, it equals to $\cos\frac{\theta}{2} - \mathrm{i}(\vec{\sigma}\cdot\vec{n})\sin\frac{\theta}{2}$.

Problem 5. Anti-unitary transformation

For an anti-unitary transformation $R = UK$, where K is complex conjugation and U is an unitary transformation. Please do the following exercises

1) please check that $R^{-1} = KU^{\dagger} = KU^{-1}$, and we can evaluate $\langle R\psi|R\phi\rangle = \langle\phi|\psi\rangle$.

2) Prove that $\langle R^{-1}\psi|R^{-1}\phi\rangle = \langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$

Problem 6. Time-reversal transformation

Please do the following exercise, where T is the time-reversal transformation.

1) From $[L_i, L_j] = \mathrm{i}\epsilon_{ijk}L_k$, derive that $T\mathrm{i}T^{-1} = -\mathrm{i}$

2) For $H = \frac{(P-\frac{e}{c}A)^2}{2m}$, what's $H^T = THT^{-1} =$?

3) If $T^2 = 1$, is there always an energy level degeneracy?

Problem 7. Parity Transformation

Please do the following exercises, where P is the parity transformation.

1) Prove that P is an unitary transformation and up to a phase factor we can always choose $P^2 = 1$. Explain the difference between T^2 and P^2.

2)Verify for momentum eigenstate $\psi_p(x,t) = e^{-\mathrm{i}px-\mathrm{i}\omega t}$, what are $\psi_p^T(x,t)$ and $\psi_p^P(x,t)$? How about angular momentum eigenstates $\psi_m(x,t) = e^{\mathrm{i}m\varphi-\mathrm{i}\omega t}$?