

The 1925 Born and Jordan paper “On quantum mechanics”

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The 1925 paper “On quantum mechanics” by M. Born and P. Jordan, and the sequel “On quantum mechanics II” by M. Born, W. Heisenberg, and P. Jordan, developed Heisenberg’s pioneering theory into the first complete formulation of quantum mechanics. The Born and Jordan paper is the subject of the present article. This paper introduced matrices to physicists. We discuss the original postulates of quantum mechanics, present the two-part discovery of the law of commutation, and clarify the origin of Heisenberg’s equation. We show how the 1925 proof of energy conservation and Bohr’s frequency condition served as the gold standard with which to measure the validity of the new quantum mechanics. © 2009 American Association of Physics Teachers.

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I. INTRODUCTION

The name “quantum mechanics” was coined by Max Born.¹ For Born and others, quantum mechanics denoted a canonical theory of atomic and electronic motion of the same level of generality and consistency as classical mechanics. The transition from classical mechanics to a true quantum mechanics remained an elusive goal prior to 1925.

Heisenberg made the breakthrough in his historic 1925 paper, “Quantum-theoretical reinterpretation of kinematic and mechanical relations.”² Heisenberg’s bold idea was to retain the classical equations of Newton but to replace the classical position coordinate with a “quantum-theoretical quantity.” The new position quantity contains information about the measurable line spectrum of an atom rather than the unobservable orbit of the electron. Born realized that Heisenberg’s kinematical rule for multiplying position quantities was equivalent to the mathematical rule for multiplying matrices. The next step was to formalize Heisenberg’s theory using the language of matrices.

The first comprehensive exposition on quantum mechanics in matrix form was written by Born and Jordan,⁴ and the sequel was written by Born, Heisenberg, and Jordan.⁵ Dirac independently discovered the general equations of quantum mechanics without using matrix theory.⁶ These papers developed a Hamiltonian mechanics of the atom in a completely new quantum (noncommutative) format. These papers ushered in a new era in theoretical physics where Hermitian matrices, commutators, and eigenvalue problems became the mathematical trademark of the atomic world. We discuss the first paper “On quantum mechanics.”⁴

This formulation of quantum mechanics, now referred to as matrix mechanics,⁷ marked one of the most intense periods of discovery in physics. The ideas and formalism behind the original matrix mechanics are absent in most textbooks. Recent articles discuss the correspondence between classical harmonics and quantum jumps,⁸ the calculational details of Heisenberg’s paper,⁹ and the role of Born in the creation of quantum theory.¹⁰ References 11–19 represent a sampling of the many sources on the development of quantum mechanics.

Given Born and Jordan’s pivotal role in the discovery of quantum mechanics, it is natural to wonder why there are no equations named after them,²⁰ and why they did not share the Nobel Prize with others.²¹ In 1933 Heisenberg wrote Born saying “The fact that I am to receive the Nobel Prize alone,

for work done in Göttingen in collaboration—you, Jordan, and I, this fact depresses me and I hardly know what to write to you. I am, of course, glad that our common efforts are now appreciated, and I enjoy the recollection of the beautiful time of collaboration. I also believe that all good physicists know how great was your and Jordan’s contribution to the structure of quantum mechanics—and this remains unchanged by a wrong decision from outside. Yet I myself can do nothing but thank you again for all the fine collaboration and feel a little ashamed.”²³

Engraved on Max Born’s tombstone is a one-line epitaph: $pq - qp = h/2\pi i$. Born composed this elegant equation in early July 1925 and called it “die verschärfte Quantenbedingung”⁴—the sharpened quantum condition. This equation is now known as the law of commutation and is the hallmark of quantum algebra.

In the contemporary approach to teaching quantum mechanics, matrix mechanics is usually introduced after a thorough discussion of wave mechanics. The Heisenberg picture is viewed as a unitary transformation of the Schrödinger picture.²⁴ How was matrix mechanics formulated in 1925 when the Schrödinger picture was nowhere in sight? The Born and Jordan paper⁴ represents matrix mechanics in its purest form.

II. BACKGROUND TO “ON QUANTUM MECHANICS”

Heisenberg’s program, as indicated by the title of his paper,² consisted of constructing quantum-theoretical relations by reinterpreting the classical relations. To appreciate what Born and Jordan did with Heisenberg’s reinterpretations, we discuss in the Appendix four key relations from Heisenberg’s paper.² Heisenberg wrote the classical and quantum versions of each relation in parallel—as formula couplets. Heisenberg has been likened to an “expert decoder who reads a cryptogram.”²⁵ The correspondence principle^{8,26} acted as a “code book” for translating a classical relation into its quantum counterpart. Unlike his predecessors who used the correspondence principle to produce specific relations, Heisenberg produced an entirely new theory—complete with a new representation of position and a new rule of multiplication, together with an equation of motion and a quantum condition whose solution determined the atomic observables (energies, frequencies, and transition amplitudes).

Matrices are not explicitly mentioned in Heisenberg's paper. He did not arrange his quantum-theoretical quantities into a table or array. In looking back on his discovery, Heisenberg wrote, "At that time I must confess I did not know what a matrix was and did not know the rules of matrix multiplication."¹⁸ In the last sentence of his paper he wrote "whether this method after all represents far too rough an approach to the physical program of constructing a theoretical quantum mechanics, an obviously very involved problem at the moment, can be decided only by a more intensive mathematical investigation of the method which has been very superficially employed here."²⁷

Born took up Heisenberg's challenge to pursue "a more intensive mathematical investigation." At the time Heisenberg wrote his paper, he was Born's assistant at the University of Göttingen. Born recalls the moment of inspiration when he realized that position and momentum were matrices.²⁸

After having sent Heisenberg's paper to the *Zeitschrift für Physik* for publication, I began to ponder about his symbolic multiplication, and was soon so involved in it...For I felt there was something fundamental behind it...And one morning, about 10 July 1925, I suddenly saw the light: Heisenberg's symbolic multiplication was nothing but the matrix calculus, well known to me since my student days from the lectures of Rosanes in Breslau.

I found this by just simplifying the notation a little: instead of $q(n, n + \tau)$, where n is the quantum number of one state and τ the integer indicating the transition, I wrote $q(n, m)$, and rewriting Heisenberg's form of Bohr's quantum condition, I recognized at once its formal significance. It meant that the two matrix products $\mathbf{pq}$ and $\mathbf{qp}$ are not identical. I was familiar with the fact that matrix multiplication is not commutative; therefore I was not too much puzzled by this result. Closer inspection showed that Heisenberg's formula gave only the value of the diagonal elements ($m = n$) of the matrix $\mathbf{pq} - \mathbf{qp}$; it said they were all equal and had the value $\hbar/2\pi i$ where $\hbar$ is Planck's constant and $i = \sqrt{-1}$. But what were the other elements ($m \neq n$)?

Here my own constructive work began. Repeating Heisenberg's calculation in matrix notation, I soon convinced myself that the only reasonable value of the nondiagonal elements should be zero, and I wrote the strange equation

$$\mathbf{pq} - \mathbf{qp} = \frac{\hbar}{2\pi i} \mathbf{1}, \quad (1)$$

where $\mathbf{1}$ is the unit matrix. But this was only a guess, and all my attempts to prove it failed.

On 19 July 1925, Born invited his former assistant Wolf-

gang Pauli to collaborate on the matrix program. Pauli declined the invitation.²⁹ The next day, Born asked his student Pascual Jordan to assist him. Jordan accepted the invitation and in a few days proved Born's conjecture that all nondiagonal elements of $\mathbf{pq} - \mathbf{qp}$ must vanish. The rest of the new quantum mechanics rapidly solidified. The Born and Jordan paper was received by the *Zeitschrift für Physik* on 27 September 1925, two months after Heisenberg's paper was received by the same journal. All the essentials of matrix mechanics as we know the subject today fill the pages of this paper.

In the abstract Born and Jordan wrote "The recently published theoretical approach of Heisenberg is here developed into a systematic theory of quantum mechanics (in the first place for systems having one degree of freedom) with the aid of mathematical matrix methods."³⁰ In the introduction they go on to write "The physical reasoning which led Heisenberg to this development has been so clearly described by him that any supplementary remarks appear superfluous. But, as he himself indicates, in its formal, mathematical aspects his approach is but in its initial stages. His hypotheses have been applied only to simple examples without being fully carried through to a generalized theory. Having been in an advantageous position to familiarize ourselves with his ideas throughout their formative stages, we now strive (since his investigations have been concluded) to clarify the mathematically formal content of his approach and present some of our results here. These indicate that it is in fact possible, starting with the basic premises given by Heisenberg, to build up a closed mathematical theory of quantum mechanics which displays strikingly close analogies with classical mechanics, but at the same time preserves the characteristic features of quantum phenomena."³¹

The reader is introduced to the notion of a matrix in the third paragraph of the introduction: "The mathematical basis of Heisenberg's treatment is the *law of multiplication* of quantum-theoretical quantities, which he derived from an ingenious consideration of correspondence arguments. The development of his formalism, which we give here, is based upon the fact that this rule of multiplication is none other than the well-known mathematical rule of *matrix multiplication*. The infinite square array which appears at the start of the next section, termed a *matrix*, is a representation of a physical quantity which is given in classical theory as a function of time. The mathematical method of treatment inherent in the new quantum mechanics is thereby characterized by the employment of *matrix analysis* in place of the usual number analysis."

The Born-Jordan paper⁴ is divided into four chapters. Chapter 1 on "Matrix calculation" introduces the mathematics (algebra and calculus) of matrices to physicists. Chapter 2 on "Dynamics" establishes the fundamental postulates of quantum mechanics, such as the law of commutation, and derives the important theorems, such as the conservation of energy. Chapter 3 on "Investigation of the anharmonic oscillator" contains the first rigorous (correspondence free) calculation of the energy spectrum of a quantum-mechanical harmonic oscillator. Chapter 4 on "Remarks on electrodynamics" contains a procedure—the first of its kind—to quantize the electromagnetic field. We focus on the material in Chap. 2 because it contains the essential physics of matrix mechanics.

III. THE ORIGINAL POSTULATES OF QUANTUM MECHANICS

Current presentations of quantum mechanics frequently are based on a set of postulates.³² The Born–Jordan postulates of quantum mechanics were crafted before wave mechanics was formulated and thus are quite different than the Schrödinger-based postulates in current textbooks. The original postulates come as close as possible to the classical-mechanical laws while maintaining complete quantum-mechanical integrity.

Section III, “The basic laws,” in Chap. 2 of the Born–Jordan paper is five pages long and contains approximately thirty equations. We have imposed a contemporary postulatory approach on this section by identifying five fundamental passages from the text. We call these five fundamental ideas “the postulates.” We have preserved the original phrasing, notation, and logic of Born and Jordan. The labeling and the naming of the postulates is ours.

Postulate 1. Position and Momentum. Born and Jordan introduce the position and momentum matrices by writing that³³

The dynamical system is to be described by the spatial coordinate $\mathbf{q}$ and the momentum $\mathbf{p}$, these being represented by the matrices

$$\begin{aligned}(\mathbf{q} &= q(nm)e^{2\pi i\nu(nm)t}, \\ (\mathbf{p} &= p(nm)e^{2\pi i\nu(nm)t}).\end{aligned}\quad (2)$$

Here the $\nu(nm)$ denote the quantum-theoretical frequencies associated with the transitions between states described by the *quantum numbers* n and m . The matrices (2) are to be Hermitian, e.g., on transposition of the matrices, each element is to go over into its complex conjugate value, a condition which should apply for all real t . We thus have

$$q(nm)q(mn) = |q(nm)|^2 \quad (3)$$

and

$$\nu(nm) = -\nu(mn). \quad (4)$$

If q is a *Cartesian* coordinate, then the expression (3) is a measure of the *probabilities* of the transitions $n \rightleftharpoons m$.

The preceding passage placed Hermitian matrices into the physics limelight. Prior to the Born–Jordan paper, matrices were rarely seen in physics.³⁴ Hermitian matrices were even stranger. Physicists were reluctant to accept such an abstract mathematical entity as a description of physical reality.

For Born and Jordan, $\mathbf{q}$ and $\mathbf{p}$ do not specify the position and momentum of an electron in an atom. Heisenberg stressed that quantum theory should focus only on the observable properties, namely the frequency and intensity of the atomic radiation and not the position and period of the electron. The quantities $\mathbf{q}$ and $\mathbf{p}$ represent position and momentum in the sense that $\mathbf{q}$ and $\mathbf{p}$ satisfy matrix equations of motion that are identical in form to those satisfied by the

position and momentum of classical mechanics. In the Bohr atom the electron undergoes periodic motion in a well defined orbit around the nucleus with a certain classical frequency. In the Heisenberg–Born–Jordan atom there is no longer an orbit, but there is some sort of periodic “quantum motion” of the electron characterized by the set of frequencies $\nu(nm)$ and amplitudes $q(nm)$. Physicists believed that something inside the atom must vibrate with the right frequencies even though they could not visualize what the quantum oscillations looked like. The mechanical properties ($\mathbf{q}, \mathbf{p}$) of the quantum motion contain complete information on the spectral properties (frequency, intensity) of the emitted radiation.

The diagonal elements of a matrix correspond to the states, and the off-diagonal elements correspond to the transitions. An important property of all dynamical matrices is that the diagonal elements are independent of time. The Hermitian rule in Eq. (4) implies the relation $\nu(nn)=0$. Thus the time factor of the n th diagonal term in any matrix is $e^{2\pi i\nu(nn)t}=1$. As we shall see, the time-independent entries in a diagonal matrix are related to the constant values of a conserved quantity.

In their purely mathematical introduction to matrices (Chap. 1), Born and Jordan use the following symbols to denote a matrix

$$\mathbf{a} = (a(nm)) = \begin{pmatrix} a(00) & a(01) & a(02) & \dots \\ a(10) & a(11) & a(12) & \\ a(20) & a(21) & a(22) & \\ \vdots & & & \ddots \end{pmatrix}. \quad (5)$$

The bracketed symbol $(a(nm))$, which displays inner elements $a(nm)$ contained within outer brackets $()$, is the shorthand notation for the array in Eq. (5). By writing the matrix elements as $a(nm)$, rather than a_{nm} , Born and Jordan made direct contact with Heisenberg’s quantum-theoretical quantities $a(n, n-\alpha)$ (see the Appendix). They wrote³⁵ “Matrix multiplication is defined by the rule ‘rows times columns,’ familiar from the ordinary theory of determinants:

$$\mathbf{a} = \mathbf{bc} \text{ means } a(nm) = \sum_{k=0}^{\infty} b(nk)c(km). \quad (6)$$

This multiplication rule was first given (for finite square matrices) by Arthur Cayley.³⁶ Little did Cayley know in 1855 that his mathematical “row times column” expression $b(nk)c(km)$ would describe the physical process of an electron making the transition $n \rightarrow k \rightarrow m$ in an atom.

Born and Jordan wrote in Postulate 1 that the quantity $|q(nm)|^2$ provides “a measure of the *probabilities* of the transitions $n \rightleftharpoons m$.” They justify this profound claim in the last chapter.³⁷ Born and Jordan’s one-line claim about transition probabilities is the only statistical statement in their postulates. Physics would have to wait several months before Schrödinger’s wave function $\Psi(x)$ and Born’s probability function $|\Psi(x)|^2$ entered the scene. Born discovered the connection between $|\Psi(x)|^2$ and position probability, and was also the first physicist (with Jordan) to formalize the connection between $|q(nm)|^2$ and the transition probability via a “quantum electrodynamic” argument.³⁸ As a pioneer statistical interpreter of quantum mechanics, it is interesting to speculate that Born might have discovered how to form a

linear superposition of the periodic matrix elements $q(nm)e^{2\pi i\nu(nm)t}$ in order to obtain another statistical object, namely the expectation value $\langle \mathbf{q} \rangle$. Early on, Born, Heisenberg, and Jordan did superimpose matrix elements,⁴⁷ but did not supply the statistical interpretation.

Postulate 2. Frequency Combination Principle. After defining $\mathbf{q}$ and $\mathbf{p}$, Born and Jordan wrote³⁹ “Further, we shall require that

$$\nu(jk) + \nu(kl) + \nu(lj) = 0. \quad (7)$$

The frequency sum rule in Eq. (7) is the fundamental constraint on the quantum-theoretical frequencies. This rule is based on the Ritz combination principle, which explains the relations of the spectral lines of atomic spectroscopy.⁴⁰ Equation (7) is the quantum analogue of the “Fourier combination principle”, $\nu(k-j) + \nu(l-k) + \nu(j-l) = 0$, where $\nu(\alpha) = \alpha\nu(1)$ is the frequency of the α th harmonic component of a Fourier series. The frequency spectrum of classical periodic motion obeys this Fourier sum rule. The equal Fourier spacing of classical lines is replaced by the irregular Ritzian spacing of quantal lines. In the correspondence limit of large quantum numbers and small quantum jumps the atomic spectrum of Ritz reduces to the harmonic spectrum of Fourier.^{8,26} Because the Ritz rule was considered an exact law of atomic spectroscopy, and because Fourier series played a vital role in Heisenberg’s analysis, it made sense for Born and Jordan to posit the frequency rule in Eq. (7) as a basic law.

One might be tempted to regard Eq. (7) as equivalent to the Bohr frequency condition, $E(n) - E(m) = h\nu(nm)$, where $E(n)$ is the energy of the stationary state n . For Born and Jordan, Eq. (7) says nothing about energy. They note that Eqs. (4) and (7) imply that there exists spectral terms W_n such that

$$h\nu(nm) = W_n - W_m. \quad (8)$$

At this postulatory stage, the term W_n of the spectrum is unrelated to the energy $E(n)$ of the state. Heisenberg emphasized this distinction between “term” and “energy” in a letter to Pauli summarizing the Born–Jordan theory.⁴¹ Born and Jordan adopt Eq. (7) as a postulate—one based solely on the observable spectral quantities $\nu(nm)$ without reference to any mechanical quantities $E(n)$. The Bohr frequency condition is not something they assume *a priori*, it is something that must be rigorously proved.

The Ritz rule insures that the nm element of any dynamical matrix (any function of $\mathbf{p}$ and $\mathbf{q}$) oscillates with the same frequency $\nu(nm)$ as the nm element of $\mathbf{p}$ and $\mathbf{q}$. For example, if the $3 \rightarrow 2$ elements of $\mathbf{p}$ and $\mathbf{q}$ oscillate at 500 MHz, then the $3 \rightarrow 2$ elements of $\mathbf{p}^2$, $\mathbf{q}^2$, $\mathbf{pq}$, $\mathbf{q}^3$, $\mathbf{p}^2 + \mathbf{q}^2$, etc. each oscillate at 500 MHz. In all calculations involving the canonical matrices $\mathbf{p}$ and $\mathbf{q}$, no new frequencies are generated. A consistent quantum theory must preserve the frequency spectrum of a particular atom because the spectrum is the spectroscopic signature of the atom. The calculations must not change the identity of the atom. Based on the rules for manipulating matrices and combining frequencies, Born and Jordan wrote that “it follows that a function $\mathbf{g}(\mathbf{pq})$ invariably takes on the form

$$\mathbf{g} = (g(nm)e^{2\pi i\nu(nm)t}) \quad (9)$$

and the matrix $(g(nm))$ therein results from identically the same process applied to the matrices $(q(nm))$, $(p(nm))$ as

was employed to find $\mathbf{g}$ from $\mathbf{q}$, $\mathbf{p}$.”⁴² Because $e^{2\pi i\nu(nm)t}$ is the universal time factor common to all dynamical matrices, they note that it can be dropped from Eq. (2) in favor of the shorter notation $\mathbf{q} = (q(nm))$ and $\mathbf{p} = (p(nm))$.

Why does the Ritz rule insure that the time factors of $\mathbf{g}(\mathbf{pq})$ are identical to the time factors of $\mathbf{p}$ and $\mathbf{q}$? Consider the potential energy function $\mathbf{q}^2$. The nm element of $\mathbf{q}^2$, which we denote by $\mathbf{q}^2(nm)$, is obtained from the elements of $\mathbf{q}$ via the multiplication rule

$$\mathbf{q}^2(nm) = \sum_k q(nk)e^{2\pi i\nu(nk)t} q(km)e^{2\pi i\nu(km)t}. \quad (10)$$

Given the Ritz relation $\nu(nm) = \nu(nk) + \nu(km)$, which follows from Eqs. (4) and (7), Eq. (10) reduces to

$$\mathbf{q}^2(nm) = \left[\sum_k q(nk)q(km) \right] e^{2\pi i\nu(nm)t}. \quad (11)$$

It follows that the nm time factor of $\mathbf{q}^2$ is the same as the nm time factor of $\mathbf{q}$.

We see that the theoretical rule for multiplying mechanical amplitudes, $a(nm) = \sum_k b(nk)c(km)$, is intimately related to the experimental rule for adding spectral frequencies, $\nu(nm) = \nu(nk) + \nu(km)$. The Ritz rule occupied a prominent place in Heisenberg’s discovery of the multiplication rule (see the Appendix). Whenever a contemporary physicist calculates the total amplitude of the quantum jump $n \rightarrow k \rightarrow m$, the steps involved can be traced back to the frequency combination principle of Ritz.

Postulate 3. The Equation of Motion. Born and Jordan introduce the law of quantum dynamics by writing⁴³

In the case of a Hamilton function having the form

$$\mathbf{H} = \frac{1}{2m}\mathbf{p}^2 + \mathbf{U}(\mathbf{q}), \quad (12)$$

we shall assume, as did Heisenberg, that the equations of motion have just the same form as in the classical theory, so that we can write:

$$\dot{\mathbf{q}} = \frac{\partial \mathbf{H}}{\partial \mathbf{p}} = \frac{1}{m}\mathbf{p}, \quad (13a)$$

$$\dot{\mathbf{p}} = -\frac{\partial \mathbf{H}}{\partial \mathbf{q}} = -\frac{\partial \mathbf{U}}{\partial \mathbf{q}}. \quad (13b)$$

This Hamiltonian formulation of quantum dynamics generalized Heisenberg’s Newtonian approach.⁴⁴ The assumption by Heisenberg and Born and Jordan that quantum dynamics looks the same as classical dynamics was a bold and deep assumption. For them, the problem with classical mechanics was not the dynamics (the form of the equations of motion), but rather the kinematics (the meaning of position and momentum).

Postulate 4. Energy Spectrum. Born and Jordan reveal the connection between the allowed energies of a conservative system and the numbers in the Hamiltonian matrix:

“The diagonal elements $H(nn)$ of $\mathbf{H}$ are interpreted, according to Heisenberg, as the *energies of the various states of the system*.”⁴⁵

This statement introduced a radical new idea into mainstream physics: calculating an energy spectrum reduces to finding the components of a diagonal matrix.⁴⁶ Although Born and Jordan did not mention the word eigenvalue in Ref. 4, Born, Heisenberg, and Jordan would soon formalize the idea of calculating an energy spectrum by solving an eigenvalue problem.⁵ The ad hoc rules for calculating a quantized energy in the old quantum theory were replaced by a systematic mathematical program.

Born and Jordan considered exclusively conservative systems for which $\mathbf{H}$ does not depend explicitly on time. The connection between conserved quantities and diagonal matrices will be discussed later. For now, recall that the diagonal elements of any matrix are independent of time. For the special case where all the non-diagonal elements of a dynamical matrix $\mathbf{g}(\mathbf{p}\mathbf{q})$ vanish, the quantity $\mathbf{g}$ is a constant of the motion. A postulate must be introduced to specify the physical meaning of the constant elements in $\mathbf{g}$.

In the old quantum theory it was difficult to explain why the energy was quantized. The discontinuity in energy had to be postulated or artificially imposed. Matrices are naturally quantized. The quantization of energy is built into the discrete row-column structure of the matrix array. In the old theory Bohr's concept of a stationary state of energy E_n was a central concept. Physicists grappled with the questions: Where does E_n fit into the theory? How is E_n calculated? Bohr's concept of the energy of the stationary state finally found a rigorous place in the new matrix scheme.⁴⁷

Postulate 5. The Quantum Condition. Born and Jordan state that the elements of $\mathbf{p}$ and $\mathbf{q}$ for any quantum mechanical system must satisfy the “quantum condition”:

$$\sum_k (p(nk)q(kn) - q(nk)p(kn)) = \frac{h}{2\pi i}. \quad (14)$$

Given the significance of Eq. (14) in the development of quantum mechanics, we quote Born and Jordan's “derivation” of this equation:

The equation

$$J = \oint p dq = \int_0^{1/\nu} p \dot{q} dt \quad (15)$$

of “classical” quantum theory can, on introducing the Fourier expansions of p and q ,

$$\begin{aligned} p &= \sum_{\tau=-\infty}^{\infty} p_{\tau} e^{2\pi i \nu \tau t}, \\ q &= \sum_{\tau=-\infty}^{\infty} q_{\tau} e^{2\pi i \nu \tau t}, \end{aligned} \quad (16)$$

be transformed into

$$1 = 2\pi i \sum_{\tau=-\infty}^{\infty} \tau \frac{\partial}{\partial J} (q_{\tau} p_{-\tau}). \quad (17)$$

The following expressions should correspond:

$$\sum_{\tau=-\infty}^{\infty} \tau \frac{\partial}{\partial J} (q_{\tau} p_{-\tau}) \quad (18)$$

with

$$\begin{aligned} &\frac{1}{h} \sum_{\tau=-\infty}^{\infty} (q(n+\tau, n) p(n, n+\tau) \\ &\quad - q(n, n-\tau) p(n-\tau, n)), \end{aligned} \quad (19)$$

where in the right-hand expression those $q(nm)$, $p(nm)$ which take on a negative index are to be set equal to zero. In this way we obtain the quantization condition corresponding to Eq. (17) as

$$\sum_k (p(nk)q(kn) - q(nk)p(kn)) = \frac{h}{2\pi i}. \quad (20)$$

This is a system of infinitely many equations, namely one for each value of n .⁴⁸

Why did Born and Jordan take the derivative of the action integral in Eq. (15) to arrive at Eq. (17)? Heisenberg performed a similar maneuver (see the Appendix). One reason is to eliminate any explicit dependence on the integer variable n from the basic laws. Another reason is to generate a differential expression that can readily be translated via the correspondence principle into a difference expression containing only transition quantities. In effect, a state relation is converted into a change-in-state relation. In the old quantum theory the Bohr–Sommerfeld quantum condition, $\oint p dq = nh$, determined how all state quantities depend on n . Such an ad hoc quantization algorithm has no proper place in a rigorous quantum theory, where n should not appear explicitly in any of the fundamental laws. The way in which $q(nm)$, $p(nm)$, $\nu(nm)$ depend on (nm) should not be artificially imposed, but should be naturally determined by fundamental relations involving only the canonical variables $\mathbf{q}$ and $\mathbf{p}$, without any explicit dependence on the state labels n and m . Equation (20) is one such fundamental relation.

In 1924 Born introduced the technique of replacing differentials by differences to make the “formal passage from classical mechanics to a ‘quantum mechanics’.”⁴⁹ This correspondence rule played an important role in allowing Born and others to develop the equations of quantum mechanics.⁵⁰ To motivate Born's rule note that the fundamental orbital frequency of a classical periodic system is equal to dE/dJ (E is energy and $J = \oint p dq$ is an action),⁵¹ whereas the spectral frequency of an atomic system is equal to $\Delta E/h$. Hence, the passage from a classical to a quantum frequency is made by replacing the derivative dE/dJ by the difference $\Delta E/h$.⁵² Born conjectured that this correspondence is valid for any quantity Φ . He wrote “We are therefore as good as forced to adopt the rule that we have to replace a classically calculated quantity, whenever it is of the form $\tau \partial \Phi / \partial J$ by the linear average or difference quotient $[\Phi(n+\tau) - \Phi(n)]/h$.”⁵³ The correspondence between Eqs. (18) and (19) follows from

Born's rule by letting Φ be $\Phi(n)=q(n,n-\tau)p(n-\tau,n)$, where $q(n,n-\tau)$ corresponds to q_τ and $p(n-\tau,n)$ corresponds to $p_{-\tau}$ or p_τ^* .

Born and Jordan remarked that Eq. (20) implies that $\mathbf{p}$ and $\mathbf{q}$ can never be finite matrices.⁵⁴ For the special case $\mathbf{p} = m\dot{\mathbf{q}}$ they also noted that the general condition in Eq. (20) reduces to Heisenberg's form of the quantum condition (see the Appendix). Heisenberg did not realize that his quantization rule was a relation between $\mathbf{pq}$ and $\mathbf{qp}$.⁵⁵

Planck's constant h enters into the theory via the quantum condition in Eq. (20). The quantum condition expresses the following deep law of nature: *All the diagonal components of $\mathbf{pq}-\mathbf{qp}$ must equal the universal constant $h/2\pi i$.*

What about the nondiagonal components of $\mathbf{pq}-\mathbf{qp}$? Born claimed that they were all equal to zero. Jordan proved Born's claim. It is important to emphasize that Postulate 5 says nothing about the nondiagonal elements. Born and Jordan were careful to distinguish the postulated statements (laws of nature) from the derivable results (consequences of the postulates). Born's development of the diagonal part of $\mathbf{pq}-\mathbf{qp}$ and Jordan's derivation of the nondiagonal part constitute the two-part discovery of the law of commutation.

IV. THE LAW OF COMMUTATION

Born and Jordan write the following equation in Sec. IV of "On quantum mechanics":

$$\mathbf{pq} - \mathbf{qp} = \frac{h}{2\pi i} \mathbf{1}. \quad (21)$$

They call Eq. (21) the "sharpened quantum condition" because it sharpened the condition in Eq. (20), which only fixes the diagonal elements, to one which fixes all the elements. In a letter to Pauli, Heisenberg referred to Eq. (21) as a "fundamental law of this mechanics" and as "Born's very clever idea."⁵⁶ Indeed, the commutation law in Eq. (21) is one of the most fundamental relations in quantum mechanics. This equation introduces Planck's constant and the imaginary number i into the theory in the most basic way possible. It is the golden rule of quantum algebra and makes quantum calculations unique. The way in which all dynamical properties of a system depend on h can be traced back to the simple way in which $\mathbf{pq}-\mathbf{qp}$ depend on h . In short, the commutation law in Eq. (21) stores information on the discontinuity, the non-commutativity, the uncertainty, and the complexity of the quantum world.

In their paper Born and Jordan proved that the off-diagonal elements of $\mathbf{pq}-\mathbf{qp}$ are equal to zero by first establishing a "diagonality theorem," which they state as follows: "If $\nu(nm) \neq 0$ when $n \neq m$, a condition which we wish to assume, then the formula $\dot{\mathbf{g}}=0$ denotes that $\mathbf{g}$ is a diagonal matrix with $g(nm)=\delta_{nm}g(nn)$."⁵⁷ This theorem establishes the connection between the structural (diagonality) and the temporal (constancy) properties of a dynamical matrix. It provided physicists with a whole new way to look at conservation principles: *In quantum mechanics, conserved quantities are represented by diagonal matrices.*⁵⁸

Born and Jordan proved the diagonality theorem as follows. Because all dynamical matrices $\mathbf{g}(\mathbf{pq})$ have the form in Eq. (9), the time derivative of $\mathbf{g}$ is

$$\dot{\mathbf{g}} = 2\pi i(\nu(nm)g(nm)e^{2\pi i\nu(nm)t}). \quad (22)$$

If $\dot{\mathbf{g}}=0$, then Eq. (22) implies the relation $\nu(nm)g(nm)=0$ for all (nm) . This relation is always true for the diagonal elements because $\nu(nn)$ is always equal to zero. For the off-diagonal elements, the relation $\nu(nm)g(nm)=0$ implies that $g(nm)$ must equal zero, because it is assumed that $\nu(nm) \neq 0$ for $n \neq m$. Thus, $\mathbf{g}$ is a diagonal matrix.

Hence, to show that $\mathbf{pq}-\mathbf{qp}$ is a diagonal matrix, Born and Jordan showed that the time derivative of $\mathbf{pq}-\mathbf{qp}$ is equal to zero. They introduced the matrix $\mathbf{d} \equiv \mathbf{pq}-\mathbf{qp}$ and expressed the time derivative of $\mathbf{d}$ as

$$\dot{\mathbf{d}} = \dot{\mathbf{p}}\mathbf{q} + \mathbf{p}\dot{\mathbf{q}} - \dot{\mathbf{q}}\mathbf{p} - \mathbf{q}\dot{\mathbf{p}}. \quad (23)$$

They used the canonical equations of motion in Eq. (13) to write Eq. (23) as

$$\dot{\mathbf{d}} = \mathbf{q} \frac{\partial \mathbf{H}}{\partial \mathbf{q}} - \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \mathbf{q} + \mathbf{p} \frac{\partial \mathbf{H}}{\partial \mathbf{p}} - \frac{\partial \mathbf{H}}{\partial \mathbf{p}} \mathbf{p}. \quad (24)$$

They next demonstrated that the combination of derivatives in Eq. (24) leads to a vanishing result⁵⁹ and say that "it follows that $\dot{\mathbf{d}}=0$ and $\mathbf{d}$ is a diagonal matrix. The diagonal elements of $\mathbf{d}$ are, however, specified by the quantum condition (20). Summarizing, we obtain the equation

$$\mathbf{pq} - \mathbf{qp} = \frac{h}{2\pi i} \mathbf{1}, \quad (25)$$

on introducing the unit matrix $\mathbf{1}$. We call Eq. (25) the 'sharpened quantum condition' and base all further conclusions on it."⁶⁰ Fundamental results that propagate from Eq. (25) include the equation of motion, $\dot{\mathbf{g}}=(2\pi i/h)(\mathbf{H}\mathbf{g}-\mathbf{g}\mathbf{H})$ (see Sec. V), the Heisenberg uncertainty principle, $\Delta p \Delta q \geq h/4\pi$, and the Schrödinger operator, $p=(h/2\pi i)d/dq$.

It is important to emphasize the two distinct origins of $\mathbf{pq}-\mathbf{qp}=(h/2\pi i)\mathbf{1}$. The diagonal part, $(\mathbf{pq}-\mathbf{qp})_{\text{diagonal}}=h/2\pi i$ is a *law*—an exact decoding of the approximate law $\oint p dq = nh$. The nondiagonal part, $(\mathbf{pq}-\mathbf{qp})_{\text{nondiagonal}}=0$, is a *theorem*—a logical consequence of the equations of motion. From a practical point of view Eq. (25) represents vital information on the line spectrum of an atom by defining a system of algebraic equations that place strong constraints on the magnitudes of $q(nm)$, $p(nm)$, and $\nu(nm)$.

V. THE EQUATION OF MOTION

Born and Jordan proved that the equation of motion describing the time evolution of any dynamical quantity $\mathbf{g}(\mathbf{pq})$ is

$$\dot{\mathbf{g}} = \frac{2\pi i}{h}(\mathbf{H}\mathbf{g} - \mathbf{g}\mathbf{H}). \quad (26)$$

Equation (26) is now often referred to as the Heisenberg equation.⁶¹ In Ref. 2 the only equation of motion is Newton's second law, which Heisenberg wrote as $\ddot{x}+f(x)=0$ (see the Appendix).

The "commutator" of mechanical quantities is a recurring theme in the Born–Jordan theory. The quantity $\mathbf{pq}-\mathbf{qp}$ lies at the core of their theory. Equation (26) reveals how the quantity $\mathbf{H}\mathbf{g}-\mathbf{g}\mathbf{H}$ is synonymous with the time evolution of $\mathbf{g}$. Thanks to Born and Jordan, as well as Dirac who established the connection between commutators and classical Poisson

brackets,⁶ the commutator is now an integral part of modern quantum theory. The change in focus from commuting variables to noncommuting variables represents a paradigm shift in quantum theory.

The original derivation of Eq. (26) is different from present-day derivations. In the usual textbook presentation Eq. (26) is derived from a unitary transformation of the states and operators in the Schrödinger picture.²⁴ In 1925, the Schrödinger picture did not exist. To derive Eq. (26) from their postulates Born and Jordan developed a new quantum-theoretical technology that is now referred to as “commutator algebra.” They began the proof by stating the following generalizations of Eq. (25):

$$\mathbf{p}^n \mathbf{q} = \mathbf{q} \mathbf{p}^n + n \frac{h}{2\pi i} \mathbf{p}^{n-1}, \quad (27)$$

$$\mathbf{q}^n \mathbf{p} = \mathbf{p} \mathbf{q}^n - n \frac{h}{2\pi i} \mathbf{q}^{n-1}, \quad (28)$$

which can readily be derived by induction. They considered Hamiltonians of the form

$$\mathbf{H} = \mathbf{H}_1(\mathbf{p}) + \mathbf{H}_2(\mathbf{q}), \quad (29)$$

where $\mathbf{H}_1(\mathbf{p})$ and $\mathbf{H}_2(\mathbf{q})$ are represented by power series

$$\begin{aligned} \mathbf{H}_1 &= \sum_s a_s \mathbf{p}^s, \\ \mathbf{H}_2 &= \sum_s b_s \mathbf{q}^s. \end{aligned} \quad (30)$$

After writing these expressions, they wrote⁶² “Formulae (27) and (28) indicate that

$$\mathbf{H} \mathbf{q} - \mathbf{q} \mathbf{H} = \frac{h}{2\pi i} \frac{\partial \mathbf{H}}{\partial \mathbf{p}}, \quad (31)$$

$$\mathbf{H} \mathbf{p} - \mathbf{p} \mathbf{H} = -\frac{h}{2\pi i} \frac{\partial \mathbf{H}}{\partial \mathbf{q}}. \quad (32)$$

Comparison with the equations of motion (13) yields

$$\dot{\mathbf{q}} = \frac{2\pi i}{h} (\mathbf{H} \mathbf{q} - \mathbf{q} \mathbf{H}), \quad (33)$$

$$\dot{\mathbf{p}} = \frac{2\pi i}{h} (\mathbf{H} \mathbf{p} - \mathbf{p} \mathbf{H}). \quad (34)$$

Denoting the matrix $\mathbf{H} \mathbf{g} - \mathbf{g} \mathbf{H}$ by $\left| \frac{\mathbf{H}}{\mathbf{g}} \right|$ for brevity, one has

$$\left| \frac{\mathbf{H}}{\mathbf{a} \mathbf{b}} \right| = \left| \frac{\mathbf{H}}{\mathbf{a}} \right| \mathbf{b} + \mathbf{a} \left| \frac{\mathbf{H}}{\mathbf{b}} \right|, \quad (35)$$

from which generally for $\mathbf{g} = \mathbf{g}(\mathbf{p}, \mathbf{q})$ one may conclude that

$$\dot{\mathbf{g}} = \frac{2\pi i}{h} \left| \frac{\mathbf{H}}{\mathbf{g}} \right| = \frac{2\pi i}{h} (\mathbf{H} \mathbf{g} - \mathbf{g} \mathbf{H}). \quad (36)$$

The derivation of Eq. (36) clearly displays Born and Jordan’s expertise in commutator algebra. The essential step to go from Eq. (27) to Eq. (31) is to note that Eq. (27) can be rewritten as a commutator-derivative relation, $\mathbf{p}^n \mathbf{q} - \mathbf{q} \mathbf{p}^n = (h/2\pi i) d\mathbf{p}^n/d\mathbf{p}$, which is equivalent to the n th term of the series representation of Eq. (31). The generalized commutation rules in Eqs. (27) and (28), and the relation between

commutators and derivatives in Eqs. (31) and (32) are now standard operator equations of contemporary quantum theory.

With the words, “Denoting the matrix $\mathbf{H} \mathbf{g} - \mathbf{g} \mathbf{H}$ by $\left| \frac{\mathbf{H}}{\mathbf{g}} \right|$,” Born and Jordan formalized the notion of a commutator and introduced physicists to this important quantum-theoretical object. The appearance of Eq. (36) in Ref. 4 marks the first printed statement of the general equation of motion for a dynamical quantity in quantum mechanics.

VI. THE ENERGY THEOREMS

Heisenberg, Born, and Jordan considered the conservation of energy and the Bohr frequency condition as universal laws that should emerge as logical consequences of the fundamental postulates. Proving energy conservation and the frequency condition was the ultimate measure of the power of the postulates and the validity of the theory.⁶³ Born and Jordan began Sec. IV of Ref. 4 by writing “The content of the preceding paragraphs furnishes the basic rules of the new quantum mechanics in their entirety. All the other laws of quantum mechanics, whose general validity is to be verified, must be *derivable* from these basic tenets. As instances of such laws to be proved, the law of energy conservation and the Bohr frequency condition primarily enter into consideration.”⁶⁴

The energy theorems are stated as follows:⁶⁵

$$\dot{\mathbf{H}} = 0 \quad (\text{energy conservation}), \quad (37)$$

$$h\nu(nm) = H(nn) - H(mm) \quad (\text{frequency condition}). \quad (38)$$

Equations (37) and (38) are remarkable statements on the temporal behavior of the system and the logical structure of the theory.⁶⁶ Equation (37) says that $\mathbf{H}$, which depends on the matrices $\mathbf{p}$ and $\mathbf{q}$ is always a constant of the motion even though $\mathbf{p} = \mathbf{p}(t)$ and $\mathbf{q} = \mathbf{q}(t)$ depend on time. In short, the t in $\mathbf{H}(\mathbf{p}(t), \mathbf{q}(t))$ must completely disappear. Equation (37) reveals the time independence of $\mathbf{H}$, and Eq. (38) specifies how $\mathbf{H}$ itself determines the time dependence of all other dynamical quantities.

Why should $\nu(nm)$, $H(nn)$, and $H(mm)$ be related? These quantities are completely different structural elements of different matrices. The parameter $\nu(nm)$ is a transition quantity that characterizes the off-diagonal, time-dependent part of $\mathbf{q}$ and $\mathbf{p}$. In contrast, $H(nn)$ is a state quantity that characterizes the diagonal, time-independent part of $\mathbf{H}(\mathbf{p}, \mathbf{q})$. It is a non-trivial claim to say that these mechanical elements are related.

It is important to distinguish between the Bohr meaning of $E_n - E_m = h\nu$ and the Born–Jordan meaning of $H(nn) - H(mm) = h\nu(nm)$. For Bohr, E_n denotes the mechanical energy of the electron and ν denotes the spectral frequency of the radiation. In the old quantum theory there exists ad hoc, semiclassical rules to calculate E_n . There did not exist any mechanical rules to calculate ν , independent of E_n and E_m . The relation between $E_n - E_m$ and ν was postulated. Born and Jordan did not postulate any connection between $H(nn)$, $H(mm)$, and $\nu(nm)$. The basic mechanical laws (law of motion and law of commutation) allow them to calculate the frequencies $\nu(nm)$ which parameterize $\mathbf{q}$ and the energies $H(nn)$ stored in $\mathbf{H}$. The theorem in Eq. (38) states that the

calculated values of the mechanical parameters $H(nn)$, $H(mm)$, and $\nu(nm)$ will always satisfy the relation $H(nn) - H(mm) = h\nu(nm)$.

The equation of motion (36) is the key to proving the energy theorems. Born and Jordan wrote “In particular, if in Eq. (36) we set $\mathbf{g}=\mathbf{H}$, we obtain

$$\dot{\mathbf{H}}=0. \quad (39)$$

Now that we have verified the energy-conservation law and recognized the matrix $\mathbf{H}$ to be diagonal [by the diagonality theorem, $\dot{\mathbf{H}}=0 \Rightarrow \mathbf{H}$ is diagonal], Eqs. (33) and (34) can be put into the form

$$h\nu(nm)q(nm) = (H(nn) - H(mm))q(nm), \quad (40)$$

$$h\nu(nm)p(nm) = (H(nn) - H(mm))p(nm), \quad (41)$$

from which the frequency condition follows.⁶⁷ Given the importance of this result, it is worthwhile to elaborate on the proof. Because the nm component of any matrix $\mathbf{g}$ is $g(nm)e^{2\pi i\nu(nm)t}$, the nm component of the matrix relation in Eq. (33) is

$$\begin{aligned} 2\pi i\nu(nm)q(nm)e^{2\pi i\nu(nm)t} \\ = \frac{2\pi i}{h} \sum_k (H(nk)q(km) \\ - q(nk)H(km))e^{2\pi i[\nu(nk)+\nu(km)]t}. \end{aligned} \quad (42)$$

Given the diagonality of $\mathbf{H}$, $H(nk)=H(nn)\delta_{nk}$ and $H(km)=H(mm)\delta_{km}$, and the Ritz rule, $\nu(nk)+\nu(km)=\nu(nm)$, Eq. (42) reduces to

$$\nu(nm) = \frac{1}{h}(H(nn) - H(mm)). \quad (43)$$

In this way Born and Jordan demonstrated how Bohr’s frequency condition, $h\nu(nm)=H(nn)-H(mm)$, is simply a scalar component of the matrix equation, $h\dot{\mathbf{q}}=2\pi i(\mathbf{H}\mathbf{q}-\mathbf{q}\mathbf{H})$. In any presentation of quantum mechanics it is important to explain how and where Bohr’s frequency condition logically fits into the formal structure.⁶⁸

According to Postulate 4, the n th diagonal element $H(nn)$ of $\mathbf{H}$ is equal to the energy of the n th stationary state. Logically, this postulate is needed to interpret Eq. (38) as the original frequency condition conjectured by Bohr. Born and Jordan note that Eqs. (8) and (38) imply that the mechanical energy $H(nn)$ is related to the spectral term W_n as follows: $W_n=H(nn)+\text{constant}$.⁶⁹

This mechanical proof of the Bohr frequency condition established an explicit connection between time evolution and energy. In the matrix scheme all mechanical quantities ($\mathbf{p}$, $\mathbf{q}$, and $\mathbf{g}(\mathbf{p}\mathbf{q})$) evolve in time via the set of factors $e^{2\pi i\nu(nm)t}$, where $\nu(nm)=(H(nn)-H(mm))/h$. Thus, all $\mathbf{g}$ -functions have the form⁷⁰

$$\mathbf{g} = (g(nm)e^{2\pi i(H(nn)-H(mm))t/h}). \quad (44)$$

Equation (44) exhibits how the difference in energy between state n and state m is the “driving force” behind the time evolution (quantum oscillations) associated with the change of state $n \rightarrow m$.

In the introduction of their paper, Born and Jordan write “With the aid of [the equations of motion and the quantum

condition], one can prove the general validity of the law of conservation of energy and the Bohr frequency relation in the sense conjectured by Heisenberg: this proof could not be carried through in its entirety by him even for the simple examples which he considered.”⁷¹ Because $\mathbf{p}$ and $\mathbf{q}$ do not commute, the mechanism responsible for energy conservation in quantum mechanics is significantly different than the classical mechanism. Born and Jordan emphasize this difference by writing “Whereas in classical mechanics energy conservation ($\dot{H}=0$) is directly apparent from the canonical equations, the same law of energy conservation in quantum mechanics, $\dot{\mathbf{H}}=0$ lies, as one can see, more deeply hidden beneath the surface. That its demonstrability from the assumed postulates is far from being trivial will be appreciated if, following more closely the classical method of proof, one sets out to prove $\mathbf{H}$ to be constant simply by evaluating $\dot{\mathbf{H}}$.”⁷²

We carry out Born and Jordan’s suggestion “to prove $\mathbf{H}$ to be constant simply by evaluating $\dot{\mathbf{H}}$ ” for the special Hamiltonian

$$\mathbf{H} = \mathbf{p}^2 + \mathbf{q}^3. \quad (45)$$

In order to focus on the energy calculus of the $\mathbf{p}$ and $\mathbf{q}$ matrices, we have omitted the scalar coefficients in Eq. (45).

If we write Eq. (45) as $\mathbf{H}=\mathbf{p}\mathbf{p}+\mathbf{q}\mathbf{q}\mathbf{q}$, calculate $\dot{\mathbf{H}}$, and use the equations of motion $\dot{\mathbf{q}}=2\mathbf{p}$, $\dot{\mathbf{p}}=-3\mathbf{q}^2$, we find⁷³

$$\dot{\mathbf{H}} = \mathbf{q}(\mathbf{p}\mathbf{q} - \mathbf{q}\mathbf{p}) + (\mathbf{q}\mathbf{p} - \mathbf{p}\mathbf{q})\mathbf{q}. \quad (46)$$

Equation (46) reveals how the value of $\mathbf{p}\mathbf{q}-\mathbf{q}\mathbf{p}$ uniquely determines the value of $\dot{\mathbf{H}}$. The quantum condition, $\mathbf{p}\mathbf{q}-\mathbf{q}\mathbf{p}=(h/2\pi i)\mathbf{1}$, reduces Eq. (46) to $\dot{\mathbf{H}}=0$. In classical mechanics the classical condition, $pq-qp=0$, is taken for granted in proving energy conservation. In quantum mechanics the condition that specifies the nonzero value of $\mathbf{p}\mathbf{q}-\mathbf{q}\mathbf{p}$ plays a nontrivial role in establishing energy conservation. This nontriviality is what Born and Jordan meant when they wrote that energy conservation in quantum mechanics “lies more deeply hidden beneath the surface.”

Proving the law of energy conservation and the Bohr frequency condition was the decisive test of the theory—the final validation of the new quantum mechanics. All of the pieces of the “quantum puzzle” now fit together. After proving the energy theorems, Born and Jordan wrote that “The fact that energy-conservation and frequency laws could be proved in so general a context would seem to us to furnish strong grounds to hope that this theory embraces truly deep-seated physical laws.”⁷⁴

VII. CONCLUSION

To put the discovery of quantum mechanics in matrix form into perspective, we summarize the contributions of Heisenberg and Born–Jordan. Heisenberg’s breakthrough consists of four quantum-theoretical reinterpretations (see the Appendix):

1. Replace the position coordinate $x(t)$ by the set of transition components $a(n, n-\alpha)e^{i\omega(n, n-\alpha)t}$.
2. Replace $x^2(t)$ with the set $\sum_{\alpha} a(n, n-\alpha)e^{i\omega(n, n-\alpha)t} a(n-\alpha, n-\beta)e^{i\omega(n-\alpha, n-\beta)t}$.
3. Keep Newton’s second law, $\ddot{x}+f(x)=0$, but replace x as before.

4. Replace the old quantum condition, $nh = \oint m\dot{x}^2 dt$, with $h = 4\pi m \sum_{\alpha} \{ |a(n+\alpha, n)|^2 \omega(n+\alpha, n) - |a(n, n-\alpha)|^2 \omega(n, n-\alpha) \}$.

The quantum mechanics of Born and Jordan consists of five postulates:

1. $\mathbf{q} = (q(nm)e^{2\pi i \nu(nm)t})$, $\mathbf{p} = (p(nm)e^{2\pi i \nu(nm)t})$,
2. $\nu(jk) + \nu(kl) + \nu(lj) = 0$,
3. $\dot{\mathbf{q}} = \partial \mathbf{H} / \partial \mathbf{p}$, $\dot{\mathbf{p}} = -\partial \mathbf{H} / \partial \mathbf{q}$,
4. $E_n = H(nn)$, and
5. $(\mathbf{pq} - \mathbf{qp})_{\text{diagonal}} = h/2\pi i$,

and four theorems

1. $(\mathbf{pq} - \mathbf{qp})_{\text{nondiagonal}} = 0$,
2. $\dot{\mathbf{g}} = (2\pi i/h)(\mathbf{Hg} - \mathbf{gH})$,
3. $\dot{\mathbf{H}} = 0$, and
4. $h\nu(nm) = H(nn) - H(mm)$.

Quantum mechanics evolved at a rapid pace after the papers of Heisenberg and Born–Jordan. Dirac’s paper was received on 7 November 1925.⁶ Born, Heisenberg,³ and Jordan’s paper was received on 16 November 1925.⁵ The first “textbook” on quantum mechanics appeared in 1926.⁷⁵ In a series of papers during the spring of 1926, Schrödinger set forth the theory of wave mechanics.⁷⁶ In a paper received June 25, 1926 Born introduced the statistical interpretation of the wave function.⁷⁷ The Nobel Prize was awarded to Heisenberg in 1932 (delayed until 1933) to Schrödinger and Dirac in 1933, and to Born in 1954.

APPENDIX: HEISENBERG’S FOUR BREAKTHROUGH IDEAS

We divide Heisenberg’s paper² into four major reinterpretations. For the most part we will preserve Heisenberg’s original notation and arguments.

Reinterpretation 1: Position. Heisenberg considered one-dimensional periodic systems. The classical motion of the system (in a stationary state labeled n) is described by the time-dependent position $x(n, t)$.⁷⁸ Heisenberg represents this periodic function by the Fourier series

$$x(n, t) = \sum_{\alpha} a_{\alpha}(n) e^{i\alpha\omega(n)t}. \quad (\text{A1})$$

Unless otherwise noted, sums over integers go from $-\infty$ to ∞ . The α th Fourier component related to the n th stationary state has amplitude $a_{\alpha}(n)$ and frequency $\alpha\omega(n)$. According to the correspondence principle, the α th Fourier component of the classical motion in the state n corresponds to the quantum jump from state n to state $n-\alpha$.^{8,26} Motivated by this principle, Heisenberg replaced the classical component $a_{\alpha}(n)e^{i\alpha\omega(n)t}$ by the transition component $a(n, n-\alpha)e^{i\omega(n, n-\alpha)t}$.⁷⁹ We could say that the Fourier harmonic is replaced by a “Heisenberg harmonic.” Unlike the sum over the classical components in Eq. (A1), Heisenberg realized that a similar sum over the transition components is meaningless. Such a quantum Fourier series could not describe the electron motion in one stationary state (n) because each term in the sum describes a transition process associated with two states (n and $n-\alpha$).

Heisenberg’s next step was bold and ingenious. Instead of

reinterpreting $x(t)$ as a *sum* over transition components, he represented the position by the *set* of transition components. We symbolically denote Heisenberg’s reinterpretation as

$$x \rightarrow \{a(n, n-\alpha)e^{i\omega(n, n-\alpha)t}\}. \quad (\text{A2})$$

Equation (A2) is the first breakthrough relation.

Reinterpretation 2: Multiplication. To calculate the energy of a harmonic oscillator, Heisenberg needed to know the quantity x^2 . How do you square a set of transition components? Heisenberg posed this fundamental question twice in his paper.⁸⁰ His answer gave birth to the algebraic structure of quantum mechanics. We restate Heisenberg’s question as “If x is represented by $\{a(n, n-\alpha)e^{i\omega(n, n-\alpha)t}\}$ and x^2 is represented by $\{b(n, n-\beta)e^{i\omega(n, n-\beta)t}\}$, how is $b(n, n-\beta)$ related to $a(n, n-\alpha)$?”

Heisenberg answered this question by reinterpreting the square of a Fourier series with the help of the Ritz principle. He evidently was convinced that quantum multiplication, whatever it looked like, must reduce to Fourier-series multiplication in the classical limit. The square of Eq. (A1) gives

$$x^2(n, t) = \sum_{\beta} b_{\beta}(n) e^{i\beta\omega(n)t}, \quad (\text{A3})$$

where the β th Fourier amplitude is

$$b_{\beta}(n) = \sum_{\alpha} a_{\alpha}(n) a_{\beta-\alpha}(n). \quad (\text{A4})$$

In the new quantum theory Heisenberg replaced Eqs. (A3) and (A4) with

$$x^2 \rightarrow \{b(n, n-\beta)e^{i\omega(n, n-\beta)t}\}, \quad (\text{A5})$$

where the $n \rightarrow n-\beta$ transition amplitude is

$$b(n, n-\beta) = \sum_{\alpha} a(n, n-\alpha) a(n-\alpha, n-\beta). \quad (\text{A6})$$

In constructing Eq. (A6) Heisenberg uncovered the symbolic algebra of atomic processes.

The logic behind the quantum rule of multiplication can be summarized as follows. Ritz’s sum rule for atomic frequencies, $\omega(n, n-\beta) = \omega(n, n-\alpha) + \omega(n-\alpha, n-\beta)$, implies the product rule for Heisenberg’s kinematic elements, $e^{i\omega(n, n-\beta)t} = e^{i\omega(n, n-\alpha)t} e^{i\omega(n-\alpha, n-\beta)t}$, which is the backbone of the multiplication rule in Eq. (A6). Equation (A6) allowed Heisenberg to algebraically manipulate the transition components.

Reinterpretation 3: Motion. Equations (A2), (A5), and (A6) represent the new “kinematics” of quantum theory—the new meaning of the position x . Heisenberg next turned his attention to the new “mechanics.” The goal of Heisenberg’s mechanics is to determine the amplitudes, frequencies, and energies from the given forces. Heisenberg noted that in the old quantum theory $a_{\alpha}(n)$ and $\omega(n)$ are determined by solving the classical equation of motion

$$\ddot{x} + f(x) = 0, \quad (\text{A7})$$

and quantizing the classical solution—making it depend on n —via the quantum condition

$$\oint m\dot{x}dx = nh. \quad (\text{A8})$$

In Eqs. (A7) and (A8) $f(x)$ is the force (per mass) function and m is the mass.

Heisenberg assumed that Newton's second law in Eq. (A7) is valid in the new quantum theory provided that the classical quantity x is replaced by the set of quantities in Eq. (A2), and $f(x)$ is calculated according to the new rules of amplitude algebra. Keeping the same form of Newton's law of dynamics, but adopting the new kinematic meaning of x is the third Heisenberg breakthrough.

Reinterpretation 4: Quantization. How did Heisenberg reinterpret the old quantization condition in Eq. (A8)? Given the Fourier series in Eq. (A1), the quantization condition, $nh = \oint m\dot{x}^2 dt$, can be expressed in terms of the Fourier parameters $a_\alpha(n)$ and $\omega(n)$ as

$$nh = 2\pi m \sum_{\alpha} |a_{\alpha}(n)|^2 \alpha^2 \omega(n). \quad (\text{A9})$$

For Heisenberg, setting $\oint p dx$ equal to an integer multiple of h was an arbitrary rule that did not fit naturally into the dynamical scheme. Because his theory focuses exclusively on transition quantities, Heisenberg needed to translate the old quantum condition that fixes the properties of the states to a new condition that fixes the properties of the transitions between states. Heisenberg believed¹⁴ that what matters is the difference between $\oint p dx$ evaluated for neighboring states: $[\oint p dx]_n - [\oint p dx]_{n-1}$. He therefore took the derivative of Eq. (A9) with respect to n to eliminate the forced n dependence and to produce a differential relation that can be reinterpreted as a difference relation between transition quantities. In short, Heisenberg converted

$$h = 2\pi m \sum_{\alpha} \alpha \frac{d}{dn} (|a_{\alpha}(n)|^2 \alpha \omega(n)) \quad (\text{A10})$$

to

$$h = 4\pi m \sum_{\alpha=0}^{\infty} \{ |a(n+\alpha, n)|^2 \omega(n+\alpha, n) - |a(n, n-\alpha)|^2 \omega(n, n-\alpha) \}. \quad (\text{A11})$$

In a sense Heisenberg's "amplitude condition" in Eq. (A11) is the counterpart to Bohr's frequency condition (Ritz's frequency combination rule). Heisenberg's condition relates the amplitudes of different lines within an atomic spectrum and Bohr's condition relates the frequencies. Equation (A11) is the fourth Heisenberg breakthrough.⁸¹

Equations (A7) and (A11) constitute Heisenberg's new mechanics. In principle, these two equations can be solved to find $a(n, n-\alpha)$ and $\omega(n, n-\alpha)$. No one before Heisenberg knew how to calculate the amplitude of a quantum jump. Equations (A2), (A6), (A7), and (A11) define Heisenberg's program for constructing the line spectrum of an atom from the given force on the electron.

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¹The name "quantum mechanics" appeared for the first time in the literature in M. Born, "Über Quantenmechanik," *Z. Phys.* **26**, 379–395 (1924).

²W. Heisenberg, "Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen," *Z. Phys.* **33**, 879–893 (1925), translated in Ref. 3, paper 12.

³*Sources of Quantum Mechanics*, edited by B. L. van der Waerden (Dover, New York, 1968).

⁴M. Born and P. Jordan, "Zur Quantenmechanik," *Z. Phys.* **34**, 858–888 (1925); English translation in Ref. 3, paper 13.

⁵M. Born, W. Heisenberg, and P. Jordan, "Zur Quantenmechanik II," *Z.*

Phys. **35**, 557–615 (1926), English translation in Ref. 3, paper 15.

⁶P. A. M. Dirac, "The fundamental equations of quantum mechanics," *Proc. R. Soc. London, Ser. A* **109**, 642–653 (1925), reprinted in Ref. 3, paper 14.

⁷The name "matrix mechanics" did not appear in the original papers of 1925 and 1926. The new mechanics was most often called "quantum mechanics." At Göttingen, some began to call it "matrix physics." Heisenberg disliked this terminology and tried to eliminate the mathematical term "matrix" from the subject in favor of the physical expression "quantum-theoretical magnitude." [Ref. 17, p. 362]. In his Nobel Lecture delivered 11 December 1933, Heisenberg referred to the two versions of the new mechanics as "quantum mechanics" and "wave mechanics." See *Nobel Lectures in Physics 1922–1941* (Elsevier, Amsterdam, 1965)].

⁸W. A. Fedak and J. J. Prentis, "Quantum jumps and classical harmonics," *Am. J. Phys.* **70**, 332–344 (2002).

⁹I. J. R. Aitchison, D. A. MacManus, and T. M. Snyder, "Understanding Heisenberg's 'magical' paper of July 1925: A new look at the calculational details," *Am. J. Phys.* **72**, 1370–1379 (2004).

¹⁰J. Bernstein, "Max Born and the quantum theory," *Am. J. Phys.* **73**, 999–1008 (2005).

¹¹S. Tomonaga, *Quantum Mechanics* (North-Holland, Amsterdam, 1962), Vol. 1.

¹²M. Jammer, *The Conceptual Development of Quantum Mechanics* (McGraw-Hill, New York, 1966).

¹³J. Mehra and H. Reichenberg, *The Historical Development of Quantum Theory* (Springer, New York, 1982), Vol. 3.

¹⁴Duncan, A. and Janssen, M., "On the verge of Umdeutung in Minnesota: Van Vleck and the correspondence principle," *Arch. Hist. Exact Sci.* **61**, 553–624 (2007).

¹⁵E. MacKinnon, "Heisenberg, models and the rise of matrix mechanics," *Hist. Stud. Phys. Sci.* **8**, 137–188 (1977).

¹⁶G. Birtwistle, *The New Quantum Mechanics* (Cambridge U.P., London, 1928).

¹⁷C. Jungnickel and R. McCormmach, *Intellectual Mastery of Nature* (University of Chicago Press, Chicago, 1986), Vol. 2.

¹⁸W. Heisenberg, "Development of concepts in the history of quantum theory," *Am. J. Phys.* **43**, 389–394 (1975).

¹⁹M. Bowen and J. Coster, "Born's discovery of the quantum-mechanical matrix calculus," *Am. J. Phys.* **48**, 491–492 (1980).

²⁰M. Born, *My Life: Recollections of a Nobel Laureate* (Taylor & Francis, New York, 1978). Born wrote that (pp. 218–219) "This paper by Jordan and myself contains the formulation of matrix mechanics, the first printed statement of the commutation law, some simple applications to the harmonic and anharmonic oscillator, and another fundamental idea: the quantization of the electromagnetic field (by regarding the components as matrices). Nowadays, textbooks speak without exception of Heisenberg's matrices, Heisenberg's commutation law and Dirac's field quantization."

²¹In 1928 Einstein nominated Heisenberg, Born, and Jordan for the Nobel Prize [See A. Pais, *Subtle Is the Lord: The Science and the Life of Albert Einstein* (Oxford U.P., New York, 1982), p. 515]. Possible explanations of why Born and Jordan did not receive the Nobel Prize are given in Ref. 10 and Ref. 22, pp. 191–193.

²²N. Greenspan, *The End of the Certain World: The Life and Science of Max Born* (Basic Books, New York, 2005).

²³See Ref. 20, p. 220.

²⁴E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1998), pp. 320–323.

²⁵Reference 11, p. 205.

²⁶N. Bohr, "On the quantum theory of line-spectra," reprinted in Ref. 3, paper 3.

²⁷Reference 3 p. 276 paper 12.

²⁸Reference 20, pp. 217–218.

²⁹Reference 20, p. 218.

³⁰Reference 3, pp. 277, paper 13.

³¹Reference 3, pp. 277–278, paper 13.

³²R. L. Liboff, *Introductory Quantum Mechanics* (Addison-Wesley, San Francisco, 2003), Chap. 3; R. Shankar, *Principles of Quantum Mechanics* (Plenum, New York, 1994), Chap. 4; C. Cohen-Tannoudji, B. Diu, and F. Lalöe, *Quantum Mechanics* (Wiley, New York, 1977), Chap. III.

³³Reference 3, p. 287, paper 13.

³⁴Reference 12, pp. 217–218.

³⁵Reference 3, p. 280, paper 13.

³⁶A. Cayley, "Sept différents mémoires d'analyse," *Mathematika* **50**, 272–

- 317 (1855); A. Cayley, "A memoir on the theory of matrices," *Philos. Trans. R. Soc. London, Ser. A* **148**, 17–37 (1858).
- ³⁷The last chapter (Bemerkungen zur Elektrodynamik) is not translated in Ref. 3. See Ref. 13, pp. 87–90 for a discussion of the contents of this section.
- ³⁸In Heisenberg's paper (see Ref. 2) the connection between $|q(nm)|^2$ and transition probability is implied but not discussed. See Ref. 3, pp. 30–32, for a discussion of Heisenberg's assertion that the transition amplitudes determine the transition probabilities. The relation between a squared amplitude and a transition probability originated with Bohr who conjectured that the squared Fourier amplitude of the classical electron motion provides a measure of the transition probability (see Refs. 8 and 26). The correspondence between classical intensities and quantum probabilities was studied by several physicists including H. Kramers, *Intensities of Spectral Lines* (A. F. Host and Sons, Copenhagen, 1919); R. Ladenburg in Ref. 3, paper 4, and J. H. Van Vleck, "Quantum principles and line spectra," *Bulletin of the National Research Council*, Washington, DC, 1926, pp. 118–153.
- ³⁹Reference 3, p. 287, paper 13.
- ⁴⁰W. Ritz, "Über ein neues Gesetz der Serienspektren," *Phys. Z.* **9**, 521–529 (1908); W. Ritz, "On a new law of series spectra," *Astrophys. J.* **28**, 237–243 (1908). The Ritz combination principle was crucial in making sense of the regularities in the line spectra of atoms. It was a key principle that guided Bohr in constructing a quantum theory of line spectra. Observations of spectral lines revealed that pairs of line frequencies combine (add) to give the frequency of another line in the spectrum. The Ritz combination rule is $\nu(nk) + \nu(km) = \nu(nm)$, which follows from Eqs. (4) and (7). As a universal, exact law of spectroscopy, the Ritz rule provided a powerful tool to analyze spectra and to discover new lines. Given the measured frequencies ν_1 and ν_2 of two known lines in a spectrum, the Ritz rule told spectroscopists to look for new lines at the frequencies $\nu_1 + \nu_2$ or $\nu_1 - \nu_2$.
- ⁴¹In the letter dated 18 September 1925 Heisenberg explained to Pauli that the frequencies ν_{ik} in the Born–Jordan theory obey the "combination relation $\nu_{ik} + \nu_{kj} = \nu_{ij}$ or $\nu_{ik} = (W_i - W_k)/h$ but naturally it is not to be assumed that W is the energy." See Ref. 3, p. 45.
- ⁴²Reference 3, p. 287, paper 13.
- ⁴³Reference 3, p. 289, paper 13.
- ⁴⁴Born and Jordan devote a large portion of Chap. 1 to developing a matrix calculus to give meaning to matrix derivatives such as $d\mathbf{q}/dt$ and $\partial\mathbf{H}/\partial\mathbf{p}$. They introduce the process of "symbolic differentiation" for constructing the derivative of a matrix with respect to another matrix. For a discussion of Born and Jordan's matrix calculus, see Ref. 13, pp. 68–71. To deal with arbitrary Hamiltonian functions, Born and Jordan formulated a more general dynamical law by converting the classical action principal, $\int L dt = \text{extremum}$, into a quantal action principal, $D(\mathbf{pq} - \mathbf{H}(\mathbf{pq})) = \text{extremum}$, where D denotes the trace (diagonal sum) of the Lagrangian matrix, $\mathbf{pq} - \mathbf{H}$. See Ref. 3, pp. 289–290.
- ⁴⁵Reference 3, p. 292. This statement by Born and Jordan appears in Sec. IV of their paper following the section on the basic laws. We have included it with the postulates because it is a deep assumption with far-reaching consequences.
- ⁴⁶In contemporary language the states labeled $n=0, 1, 2, 3, \dots$ in Heisenberg's paper and the Born–Jordan paper are exact stationary states (eigenstates of $\mathbf{H}$). The Hamiltonian matrix is automatically a diagonal matrix with respect to this basis.
- ⁴⁷Although Heisenberg, Born, and Jordan made the "energy of the state" and the "transition between states" rigorous concepts, it was Schrödinger who formalized the concept of the "state" itself. It is interesting to note that "On quantum mechanics II" by Born, Heisenberg, and Jordan was published before Schrödinger and implicitly contains the first mathematical notion of a quantum state. In this paper (Ref. 3, pp. 348–353), each Hermitian matrix $\mathbf{a}$ is associated with a "bilinear form" $\sum_{nm} a(nm)x_n x_m^*$. Furthermore, they identified the "energy spectrum" of a system with the set of "eigenvalues" W in the equation $Wx_k - \sum_l H(kl)x_l = 0$. In present-day symbolic language the bilinear form and eigenvalue problem are $\langle\P|\mathbf{a}|\Psi\rangle$ and $\mathbf{H}|\Psi\rangle = \mathbf{W}|\Psi\rangle$, respectively, where the variables x_n are the expansion coefficients of the quantum state $|\Psi\rangle$. At the time, they did not realize the physical significance of their eigenvector $(x_1, x_2, \dots)$ as representing a stationary state.
- ⁴⁸Reference 3, pp. 290–291, paper 13.
- ⁴⁹This is the sentence from Born's 1924 paper (See Ref. 1) where the name "quantum mechanics" appears for the first time in the physics literature [Ref. 3, p. 182].
- ⁵⁰See the chapter "The transition to quantum mechanics" in Ref. 12, pp. 181–198 for applications of "Born's correspondence rule." The most important application was deriving Kramer's dispersion formula. See Ref. 3, papers 6–10 and Ref. 14.
- ⁵¹Reference 11, pp. 144–145.
- ⁵²The exact relation between the orbital frequency and the optical frequency is derived as follows. Consider the transition from state n of energy $E(n)$ to state $n-\tau$ of energy $E(n-\tau)$. In the limit $n \gg \tau$, that is, large "orbit" and small "jump," the difference $E(n) - E(n-\tau)$ is equal to the derivative $\tau dE/dn$. Given the old quantum condition $J = nh$, it follows that $dE/dn = h dE/dJ$. Thus for $n \gg \tau$ and $J = nh$, we have the relation $[E(n) - E(n-\tau)]/h = \tau dE/dJ$, or equivalently, $\nu(n, n-\tau) = \tau \nu(n)$. This relation proves an important correspondence theorem: In the limit $n \gg \tau$, the frequency $\nu(n, n-\tau)$ associated with the quantum jump $n \rightarrow n-\tau$ is equal to the frequency $\tau \nu(n)$ associated with the τ th harmonic of the classical motion in the state n . See Refs. 8 and 26.
- ⁵³Reference 3, p. 191, paper 7.
- ⁵⁴Suppose that the number of states is finite and equal to the integer N . Then, according to Eq. (20), the diagonal sum (trace) of $\mathbf{pq} - \mathbf{qp}$ would be $D(\mathbf{pq} - \mathbf{qp}) = Nh/2\pi i$. This nonzero value of the trace contradicts the purely mathematical relation $D(\mathbf{pq} - \mathbf{qp}) = 0$, which must be obeyed by all finite matrices.
- ⁵⁵Heisenberg interview quoted in Ref. 12, p. 281, footnote 45.
- ⁵⁶Reference 17, p. 361.
- ⁵⁷Reference 3, p. 288, paper 13. The name "Diagonality theorem" is ours. The condition $\nu(nm) \neq 0$ when $n \neq m$ implies that the system is nondegenerate.
- ⁵⁸In contemporary language a conserved quantity is an operator that commutes with the Hamiltonian operator $\mathbf{H}$. For such commuting operators there exists a common set of eigenvectors. In the energy eigenbasis that underlies the Born–Jordan formulation, the matrices representing $\mathbf{H}$ and all conserved quantities are automatically diagonal.
- ⁵⁹Born and Jordan's proof that Eq. (24) vanishes is based on a purely mathematical property of "symbolic differentiation" discussed in Sec. II of their paper (See Ref. 4). For a separable Hamiltonian of the form $\mathbf{H} = \mathbf{p}^2/2m + \mathbf{U}(\mathbf{q})$, the proof is simpler. For this case Eq. (24) becomes $\dot{\mathbf{d}} = \mathbf{q}(\partial\mathbf{U}/\partial\mathbf{q}) - (\partial\mathbf{U}/\partial\mathbf{q})\mathbf{q} + \mathbf{p}(\mathbf{p}/m) - (\mathbf{p}/m)\mathbf{p}$. Because $\mathbf{p}$ and $\mathbf{q}$ are separated in this expression, we do not have to consider the inequality $\mathbf{pq} \neq \mathbf{qp}$. The expression reduces to $\dot{\mathbf{d}} = 0$.
- ⁶⁰Reference 3, p. 292, paper 13. In Ref. 4, Born and Jordan refer to $\mathbf{pq} - \mathbf{qp} = (h/2\pi i)\mathbf{1}$ as the "verschärfte Quantenbedingung," which has been translated as "sharpened quantum condition" (Ref. 13, p. 77), "stronger quantum condition" (Ref. 3, p. 292), and "exact quantum condition" (Ref. 12, p. 220).
- ⁶¹J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, San Francisco, 1994), pp. 83–84; A. Messiah, *Quantum Mechanics* (J Wiley, New York, 1958), Vol. I, p. 316.
- ⁶²Reference 3, p. 293, paper 13.
- ⁶³Proving the frequency condition—the second general principle of Bohr—was especially important because this purely quantal condition was generally regarded as a safely established part of physics. Prior to Born and Jordan's mechanical proof of the frequency condition, there existed a "thermal proof" given by Einstein in his historic paper, "On the quantum theory of radiation," *Phys. Z.* **18**, 121 (1917), translated in Ref. 3, pp. 63–77. In this paper Einstein provides a completely new derivation of Planck's thermal radiation law by introducing the notion of transition probabilities (A and B coefficients). Bohr's frequency condition emerges as the condition necessary to reduce the Boltzmann factor $\exp[(E_n - E_m)/kT]$ in Einstein's formula to the "Wien factor" $\exp(h\nu/kT)$ in Planck's formula.
- ⁶⁴Reference 3, p. 291, paper 13.
- ⁶⁵Reference 3, pp. 291–292, paper 13. Born and Jordan do not refer to the consequences in Eqs. (37) and (38) as theorems. The label "Energy theorems" is ours.
- ⁶⁶Instead of postulating the equations of motion and deriving the energy theorems, we could invert the proof and postulate the energy theorems and derive the equations of motion. This alternate logic is mentioned in Ref. 3, p. 296 and formalized in Ref. 5 (Ref. 3, p. 329). Also see J. H. Van Vleck, "Note on the postulates of the matrix quantum dynamics," *Proc. Natl. Acad. Sci. U.S.A.* **12**, 385–388 (1926).
- ⁶⁷Reference 3, pp. 293–294, paper 13. The proof of the energy theorems was based on separable Hamiltonians defined in Eq. (29). To generalize the proof Born and Jordan consider more general Hamiltonian functions

$\mathbf{H}(\mathbf{p}\mathbf{q})$ and discover the need to symmetrize the functions. For example, for $\mathbf{H}^*=\mathbf{p}^2\mathbf{q}$, it does not follow that $\dot{\mathbf{H}}^*=0$. However, they note that $\mathbf{H}=(\mathbf{p}^2\mathbf{q}+\mathbf{qp}^2)/2$ yields the same equations of motion as $\mathbf{H}^*$ and also conserves energy, $\dot{\mathbf{H}}=0$. The symmetrization rule reflects the noncommutativity of $\mathbf{p}$ and $\mathbf{q}$.

⁶⁸In the Heisenberg, Born–Jordan approach the transition components of the “matter variables” $\mathbf{q}$ and $\mathbf{p}$ are simply assumed to oscillate in time with the radiation frequencies. In contemporary texts a rigorous proof of Bohr’s frequency condition involves an analysis of the interaction between matter and radiation (radiative transitions) using time-dependent perturbation theory. See Ref. 24, Chap. 19.

⁶⁹Reference 3, p. 292, paper 13.

⁷⁰Using the language of state vectors and bra-kets, the matrix element of an operator $\mathbf{g}$ is $g_{nm}(t)=\langle\Psi_n(t)|\mathbf{g}|\Psi_m(t)\rangle$, where the energy eigenstate is $|\Psi_n(t)\rangle=\exp(-2\pi iE_nt/\hbar)|\Psi_n(0)\rangle$. This Schrödinger element is equivalent to the Born–Jordan element in Eq. (44).

⁷¹Reference 3, p. 279, paper 13. Heisenberg was able to demonstrate energy conservation and Bohr’s frequency condition for two systems (anharmonic oscillator and rotator). The anharmonic oscillator analysis was limited to second-order perturbation theory.

⁷²Reference 3. Born and Jordan do not pursue this direct method of proof noting that for the most general Hamiltonians the calculation “becomes so exceedingly involved that it seems hardly feasible.” (Ref. 3, p. 296). In a footnote on p. 296, they note that for the special case $\mathbf{H}=\mathbf{p}^2/2m+U(\mathbf{q})$, the proof can be carried out immediately. The details of this proof can be found in Ref. 73.

⁷³J. J. Prentis and W. A. Fedak, “Energy conservation in quantum mechanics,” *Am. J. Phys.* **72**, 580–590 (2004).

⁷⁴Reference 3, p. 296, paper 13.

⁷⁵M. Born, *Problems of Atomic Dynamics* (MIT Press, Cambridge, 1970).

⁷⁶E. Schrödinger, *Collected Papers on Wave Mechanics* (Chelsea, New York, 1978).

⁷⁷M. Born, “Zur Quantenmechanik der Stoßvorgänge,” *Z. Phys.* **37**, 863–867 (1926).

⁷⁸Heisenberg’s “classical” quantity $x(n,t)$ is the classical solution $x(t)$ of Newton’s equation of motion subject to the old quantum condition $\oint m\dot{x}dx=n\hbar$. For example, given the purely classical position function $x(t)=a\cos\omega t$ of a harmonic oscillator, the condition $\oint m\dot{x}^2dt=n\hbar$ quantizes the amplitude, making a depend on n as follows: $a(n)=\sqrt{n\hbar/\pi m\omega}$. Thus, the motion of the harmonic oscillator in the stationary state n is described by $x(n,t)=\sqrt{n\hbar/\pi m\omega}\cos\omega t$.

⁷⁹The introduction of transition components $a(n,n-\tau)e^{i\omega(n,n-\tau)t}$ into the formalism was a milestone in the development of quantum theory. The one-line abstract of Heisenberg’s paper reads “The present paper seeks to establish a basis for theoretical quantum mechanics founded exclusively upon relationships between quantities which in principle are observable” (Ref. 3, p. 261). For Heisenberg, the observable quantities were $a(n,n-\tau)$ and $\omega(n,n-\tau)$, that is, the amplitudes and the frequencies of the spectral lines. Prior to 1925, little was known about transition amplitudes. There was a sense that Einstein’s transition probabilities were related to the squares of the transition amplitudes. Heisenberg made the transition amplitudes (and frequencies) the central quantities of his theory. He discovered how to manipulate them, relate them, and calculate their values.

⁸⁰Reference 3, pp. 263–264, paper 12.

⁸¹Heisenberg notes (Ref. 3, p. 268, paper 12) that Eq. (A11) is equivalent to the sum rule of Kuhn and Thomas (Ref. 3, paper 11). For a discussion of Heisenberg’s development of the quantum condition, see Mehra and H. Reichenberg, *The Historical Development of Quantum Theory* (Springer, New York, 1982), Vol. 2, pp. 243–245, and Ref. 14.

SCIENTIFIC APTITUDE AND AUTISM

There’s even some evidence that scientific abilities are associated with traits characteristic of autism, the psychological disorder whose symptoms include difficulties in social relationships and communication, or its milder version, Asperger syndrome. One recent study, for instance, examined different groups according to the Autism-Spectrum Quotient test, which measures autistic traits. Scientists scored higher than nonscientists on this test, and within the sciences, mathematicians, physical scientists, and engineers scored higher than biomedical scientists.

Sidney Perkowitz, *Hollywood Science: Movies, Science, and the End of the World* (Columbia University Press, 2007), p. 170.