

Electron spin g_s and Thomas precession

{ Anomalous Zeeman effect and fine structure

$$\Rightarrow g_s = 2$$

{ Spin-orbit coupling - factor of 2 puzzle.

$$\{ U = - \mu_B g_s \vec{S} \cdot \vec{B} + \frac{g_s - 1}{2m^2 c^2} (\vec{L} \cdot \vec{S}) \left(\frac{1}{r} \frac{dV}{dr} \right)$$

$g_s - 1 \leftarrow$ Thomas precession

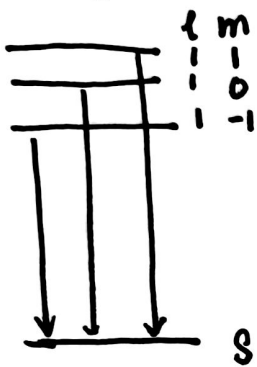
$\downarrow$
relativistic transformation $\vec{B}' = \frac{\vec{v}}{c} \times \vec{E}$

$$\vec{\omega}_T = \frac{\gamma^2}{1 + \gamma} \frac{\vec{a} \times \vec{v}}{c^2}$$

{ The dominance of Thomas precession in nuclear spin-orbit coupling.

* Zeeman effect

The atomic spectral line splits in the external B-field. If we completely neglect electron spin, only consider orbital magnetic moment, according to $\Delta l_z = \pm 1, 0$, there would only be 3-splitting lines.



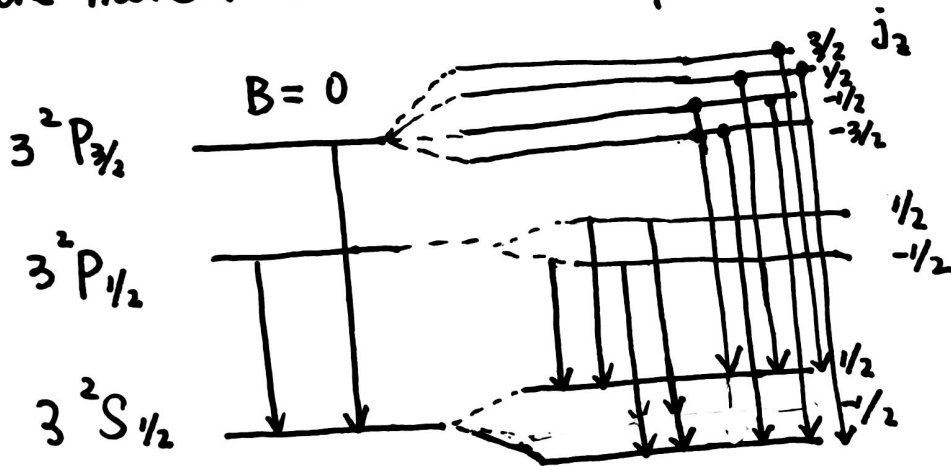
$$\vec{\mu}_L = -\frac{e}{2mc} \vec{L} = -\frac{\mu_B}{\hbar} \vec{L}, \text{ where } \mu_B = \frac{e\hbar}{2mc}$$

$$H = -\vec{B} \cdot \vec{\mu}_L$$

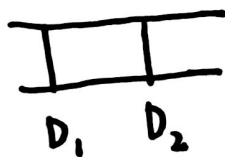


$$\mu_L = \frac{I \cdot A}{c} = \frac{e}{2\pi r} \frac{\pi r^2}{c} = \frac{e r v}{2c} = \frac{e}{2mc} \underbrace{m v r}_{\hbar} \hat{L}$$

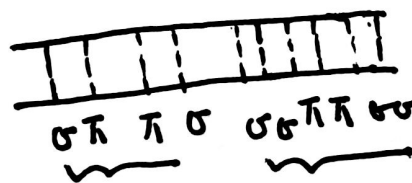
However, experimentally it is often observed that there are more than three lines. For example, the Na spectral line:



$2s+1$
 L_j



Yellow doublet



4 components

6 components

(2)

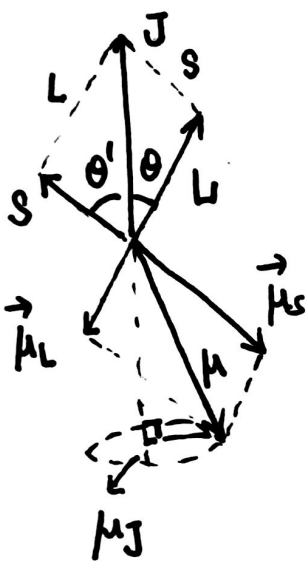
how to explain this anomalous Zeeman splittings?

Uhlenbeck and Goudsmit proposed that electron also possesses spin S . The total electron magnetic moment

$\vec{\mu} = -\frac{\mu_0}{\hbar} (\vec{L} + g_s \vec{S})$, and $\vec{S}$ contribute differently to magnetic moment $\vec{\mu}$.

The total angular momentum $\vec{J} = \vec{L} + \vec{S}$ is conserved.

Only $\vec{\mu}$'s projection along the direction of $\vec{J}$ averages non-zero



$$\mu_J = -\frac{e}{2mc} \frac{1}{\hbar} [\cos \theta \cdot L + g_s \cos \theta' \cdot S]$$

$$\left. \begin{aligned} S^2 &= J^2 + L^2 - 2JL \cos \theta \\ \Rightarrow L \cos \theta &= \frac{J^2 + L^2 - S^2}{2J} \\ L^2 &= J^2 + S^2 - 2JS \cos \theta' \\ S \cos \theta' &= \frac{J^2 + S^2 - L^2}{2J} \end{aligned} \right\} \text{weak B field}$$

$$\Rightarrow \mu_J = -\mu_0 \frac{1}{\hbar} \frac{1}{2J} [J^2 + L^2 - S^2 + g_s (J^2 + S^2 - L^2)]$$

$$= -\mu_0 \frac{J}{\hbar} \frac{1}{2J^2} \left\{ (1+g_s)J^2 + (g_s-1)(-L^2+S^2) \right\}$$

$$\vec{\mu}_J = -\mu_0 \frac{\vec{J}}{\hbar} \left\{ 1 + \frac{1}{2J^2} (g_s-1) [J^2 - L^2 + S^2] \right\}$$

Taking into account the quantum nature

②

$$J^2 \rightarrow J(J+1), \quad L^2 \rightarrow L(L+1), \quad S^2 \rightarrow S(S+1)$$

$$\vec{\mu}_J = -\frac{\mu_B}{\hbar} \vec{J} \left[1 + (g_s - 1) \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} \right]$$

$$\Delta E_{j j_z} = -\vec{B} \cdot \vec{\mu}_J = \frac{\mu_B B}{\hbar} j_z \hbar \cdot g_J = g_J \mu_B j_z$$

It can be derived exactly in QM class.

For Na $^2S_{1/2}$ state, $S=1/2$, ~~$L=0$~~ $L=0$, $J=S$

$$g_{\text{low}} = g_s$$

For $^2P_{1/2}$ state, $S=1/2$, $L=1$, $J=1/2$

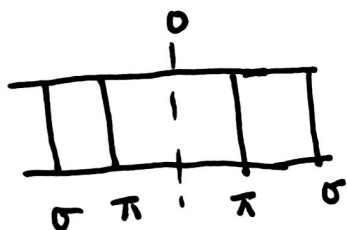
$$g_{\text{up}} = 1 + (g_s - 1) \frac{\frac{3}{4} \times 2 - 2}{\frac{3}{4} \times 2} = 1 - \frac{1}{3}(g_s - 1)$$

Hence

$$\frac{\Delta E}{\mu_B B} = \pm \frac{g_{\text{up}} + g_{\text{low}}}{2}, \quad \pm \frac{g_{\text{up}} - g_{\text{low}}}{2} \rightarrow \pi$$

$\downarrow$
 σ

In the Na spectrum $^2P_{1/2} \rightarrow ^2S_{1/2}$ transition



The ratio of shifts $-4/3 : -2/3 : 2/3 : 4/3$

$$\Rightarrow \frac{g_{\text{up}} + g_{\text{low}}}{g_{\text{up}} - g_{\text{low}}} = \frac{2}{1} \Rightarrow \frac{g_{\text{up}}}{g_{\text{low}}} = \frac{3}{1}$$

$$\Rightarrow , g_{\text{up}} = 6/5 \text{ and } g_{\text{low}} = g_s = 2/5$$

④

This does not sound right!

we could also try $\frac{g_{up} + g_{low}}{g_{low} - g_{up}} = \frac{2}{1} \Rightarrow \frac{g_{low}}{g_{up}} = \frac{3}{1}$

$$\Rightarrow g_s / \frac{4}{3} - \frac{g_s}{3} = 3 \Rightarrow g_s = 4 - g_s \Rightarrow g_s = g_{low} = ?$$

$$g_{up} = \frac{2}{3}.$$

Then we can compare the shift of the σ -line.

$$\frac{g_{up} + g_{low}}{2} \cdot \mu_B B = \frac{4}{3} \mu_B B \quad \checkmark \quad (\text{match experiments})$$

$$\left\{ \frac{4}{5} \mu_B B \right.$$

$\Rightarrow$ electron spin Lande factor $g_s = 2!$

② spin-orbit coupling

⑤

The Na yellow doublet splitting is due to SO coupling.

According to the argument later, the SO coupling can be viewed as the relativistic transformation of the nuclear electric field. the result of It would lead to

$$H_{so} = \frac{g_s}{2m^2c^2} (\vec{S} \cdot \vec{L}) \left(\frac{1}{r} \frac{dV}{dr} \right).$$

For Na $3^2S_{1/2}$ state, it's very difficult to calculate $\frac{1}{r} \frac{dV}{dr}$ due to the ionic core. Hence, it's not clear how to check whether the above expression is correct or not.

Hence, we can check the 2p state of H-atom. There is only one electron

$$V(r) = \frac{e^2}{r} \quad \frac{1}{r} \frac{dV}{dr} = \frac{1}{r} \left(-\frac{e^2}{r^2} \right) = -\frac{e^2}{r^3}$$

$$\text{For } n=2, l=1 \quad \langle r^{-3} \rangle_{2p} = \frac{1}{24a_0^3} \leftarrow \text{based hydrogen atom WF}$$

$$\langle \vec{L} \cdot \vec{S} \rangle = \langle J^2 - L^2 - S^2 \rangle / 2$$

$$\langle \vec{L} \cdot \vec{S} \rangle_{j=3/2} - \langle \vec{L} \cdot \vec{S} \rangle_{j=1/2} = \frac{1}{2} \left(\frac{3}{2} \cdot \frac{5}{2} - \frac{1}{2} \cdot \frac{3}{2} \right) = \frac{3}{2} \hbar^2$$

$$\Rightarrow \Delta E_{2p} = \frac{g_s}{2m^2c^2} \cdot \frac{3}{2} \hbar^2 \cdot \frac{e^2}{24a_0^3} = \frac{g_s}{32} \frac{\hbar^2}{m^2c^2} \frac{e^2}{a_0^3} = \frac{mc^2 \alpha^4 g_s}{32}$$

The splitting measured is actually only 1/2 of this value!

§ Thomas precession

Background: Uhlenbeck & Goudsmit proposed the existence of electron spin. By assign the Lande factor in $\vec{\mu} = g \frac{e}{2mc} \vec{S}$ as $g=2$, then the anomalous Zeeman effect can be explained. But the spectrum splitting due to fine structure is too big!

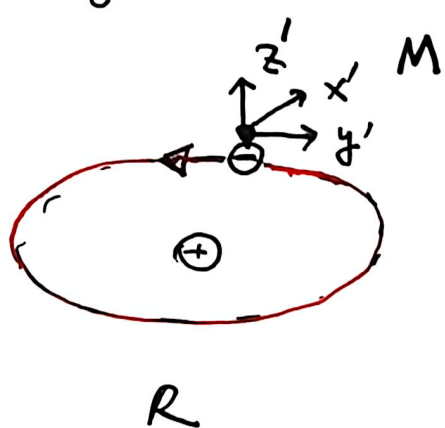
Suppose we are in co-moving frame ^M with electron

$$\left(\frac{d\vec{S}}{dt} \right)_{\text{co-moving frame} = M} = \vec{\mu} \times \vec{B}'$$

where $\vec{B}'$ is the magnetic inductance in the co-moving frame M. Then we have $\vec{B}' \simeq \gamma \left(\vec{B} - \frac{\vec{v}}{c} \times \vec{E} \right)$

$\vec{B}$ and $\vec{E}$ are the EM field in the frame that the nucleus is at rest. $\vec{v}$ is electron's velocity in frame R.
 denoted as R.

Hence $\left(\frac{d\vec{S}}{dt} \right)_M = \vec{\mu} \times \left(\vec{B} - \frac{\vec{v}}{c} \times \vec{E} \right)$



R

This EOM can be derived via the energy of

$$U' = -\vec{\mu} \cdot (\vec{B} - \frac{\vec{v}}{c} \times \vec{E})$$

(here $e < 0$

electron charge)

$$e \vec{E} = -\hat{r} \cdot \frac{dV}{dr} = -\frac{\vec{r}}{r} \frac{dV}{dr}$$

$$\Rightarrow U' = -\frac{ge}{2mc} \vec{S} \cdot (\vec{B} + \frac{\vec{v}}{c} \times \vec{r} \frac{1}{r} \frac{dV}{dr})$$

$$= -\frac{ge}{2mc} \vec{S} \cdot \vec{B} + \frac{g}{2m^2c^2} (\vec{S} \cdot \vec{L}) \left(\frac{1}{r} \frac{dV}{dr} \right)$$

$\vec{L} = m \vec{r} \times \vec{v}$

spin-orbit coupling

The sign of $\vec{L} \cdot \vec{S}$ term is

correct. However, its magnitude is twice larger than the correct value.

Thomas pointed out if the M-frame is rotating, we will have

$$\left(\frac{d\vec{S}}{dt} \right)_F = \left(\frac{d\vec{S}}{dt} \right)_M + \vec{\omega}_T \times \vec{S}$$

$$= \vec{S} \times \left(\frac{ge}{2mc} \vec{B}' - \vec{\omega}_T \right)$$

This corresponds to the coupling energy

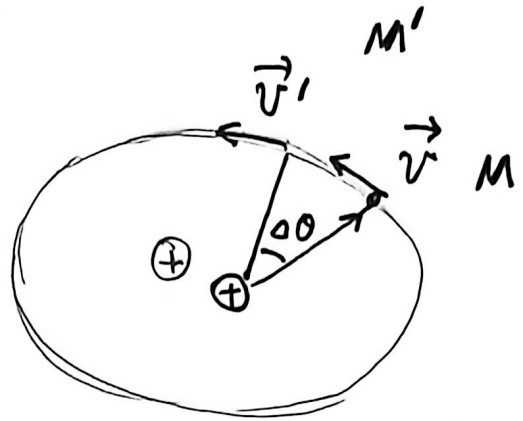
$$U = U' + \vec{S} \cdot \vec{\omega}_T$$

(3)

How come M is rotating?

Relative to F , the velocity of M is changing.

Here's the logic:



M : relative to F , boost along the direction of $\vec{v}$: $K(\vec{v})$

M' relative to M , is a pure boost
(since force is centripetal, no-torque in M) denoted as $K(\Delta\vec{u})$

Then M' relative to F 's velocity $\vec{v}'$, but

$$K(\vec{v}') \neq K(\Delta\vec{u}) K(\vec{v}), \text{ but}$$

$$K(\vec{v}') = R(\Delta\Omega) K(\Delta\vec{u}) K(\vec{v}), \text{ where } R(\Delta\Omega) \text{ is}$$

a rotation around $\hat{z}$ -axis.

We consider $\vec{v}' \rightarrow \vec{v}$ such that $\Delta\vec{u}, \Delta\Omega$ are infinitesimal. We calculate

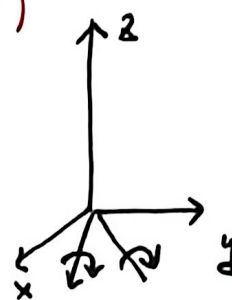
$$K(\vec{v}') K^{-1}(\vec{v}) \text{ and express it as an infinitesimal boost and a rotation } R(\Delta\Omega) K(\Delta\vec{u})$$

$$K(\vec{v}') K^{-1}(\vec{v}) \approx : R(\Delta\Omega) K(\Delta\vec{u})$$

$$\text{and } \omega_T = \Delta\Omega / \Delta t$$

★ Warm-up: Euclidean rotation (spinor Rep)

Consider a rotation in the plane of $\hat{r} \times \hat{z}$, where $\hat{r}$ lies in the xy -plane. Rotation angle is r , define $\vec{r} = r \cdot \hat{r}$.



$$U_{\hat{r}' \times \hat{z}}(r') = \cos \frac{r'}{2} - i \sin \frac{r'}{2} \vec{\sigma} \cdot (\hat{r}' \times \hat{z})$$

$$U_{\hat{r} \times \hat{z}}(r) = \cos \frac{r}{2} - i \sin \frac{r}{2} \vec{\sigma} \cdot (\hat{r} \times \hat{z})$$

For simplicity, we set $r = r'$, and $\Delta \vec{r} = \vec{r}' - \vec{r}$.

Then

$$U_{\hat{r}' \times \hat{z}}(r) U_{\hat{r} \times \hat{z}}^{-1}(r) = \cos \frac{r}{2} \cos \frac{r}{2} + \sin \frac{r}{2} \sin \frac{r}{2} \underbrace{\vec{\sigma} \cdot (\hat{r}' \times \hat{z}) \cdot \vec{\sigma} \cdot (\hat{r} \times \hat{z})}_{\vec{\sigma} \cdot (\hat{r}' \times \hat{z}) \cdot \vec{\sigma} \cdot (\hat{r} \times \hat{z})} + i \cos \frac{r}{2} \sin \frac{r}{2} \vec{\sigma} \cdot (\hat{r} \times \hat{z}) - i \cos \frac{r}{2} \sin \frac{r}{2} \vec{\sigma} \cdot (\hat{r}' \times \hat{z})$$

$$[\vec{\sigma} \cdot (\hat{r}' \times \hat{z})][\vec{\sigma} \cdot (\hat{r} \times \hat{z})] = (\hat{r}' \times \hat{z})(\hat{r} \times \hat{z}) + i \vec{\sigma} \cdot [(\hat{r}' \times \hat{z}) \times (\hat{r} \times \hat{z})]$$

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$$

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{a} [\vec{a} \cdot (\vec{b} \times \vec{c})]$$

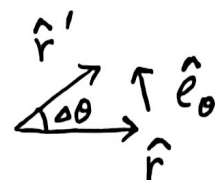
$$\Rightarrow [\vec{\sigma} \cdot (\hat{r}' \times \hat{z})][\vec{\sigma} \cdot (\hat{r} \times \hat{z})] = \hat{r}' \cdot \hat{r} + i \vec{\sigma} \cdot (\hat{r}' \times \hat{r})$$

$$\Rightarrow U_{\hat{r}' \times \hat{z}}(r) U_{\hat{r} \times \hat{z}}^{-1}(r) \simeq \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2} [1 + i \vec{\sigma}_z \cdot (\hat{r}' \times \hat{r})]$$

$$+ i \cos \frac{r}{2} \sin \frac{r}{2} \vec{\sigma} \cdot [(\hat{r} - \hat{r}') \times \hat{z}]$$

$$= 1 - i \sin^2 \frac{r}{2} \frac{1}{r^2} (\vec{r} \times \vec{r}') \cdot \vec{\sigma}_z + i \sin \frac{r}{2} [(\hat{r}' - \hat{r}) \times \hat{z}] \cdot \vec{\sigma}$$

set $\hat{r}' - \hat{r} = \Delta\theta \cdot \hat{e}_\theta$, $\hat{e}_\theta \times \hat{z} = \hat{r}$



$$\frac{1}{r^2} (\vec{r} \times \vec{r}') = \Delta\theta \cdot \hat{z}$$

$$U_{\hat{r}' \times \hat{z}}(r) U_{\hat{r} \times \hat{z}}^{-1}(r) \simeq 1 - i \sin^2 \frac{r}{2} \cdot \Delta\theta \sigma_z + i \frac{\sin r \Delta\theta}{2} (\vec{\sigma} \cdot \hat{r})$$

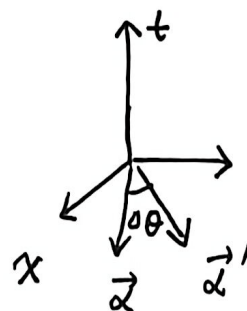
$$\simeq U_{\vec{\sigma} \cdot \hat{r}}(-\sin r \Delta\theta) U_{\hat{z}}[(1 - \cos r) \cdot \Delta\theta]$$

(*) Next we will apply the similar method to

Lorentz boost

Fundamental spinor

$$U(\vec{\theta}, \vec{\alpha}) = e^{-(i\vec{\theta} + \vec{\alpha}) \cdot \frac{\vec{\sigma}}{2}}$$



Boost $K(\vec{\alpha}) = e^{-\vec{\alpha} \cdot \frac{\vec{\sigma}}{2}} = \cosh \frac{\alpha}{2} - \sinh \frac{\alpha}{2} (\hat{\alpha} \cdot \vec{\sigma})$

For convenience, we switch to rapidity

→ Vector Rep for $\alpha = \alpha \hat{z}$

$$\begin{pmatrix} \cosh \alpha & 0 & 0 & -\sinh \alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh \alpha & 0 & 0 & \cosh \alpha \end{pmatrix} \rightarrow \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}$$

$$\Rightarrow \boxed{\tanh \alpha = \beta}$$

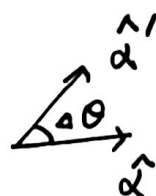
$$K(\vec{\alpha}') K^{-1}(\vec{\alpha}) = \left[\cosh \frac{\alpha}{2} - \sinh \frac{\alpha}{2} (\hat{\alpha}' \cdot \vec{\sigma}) \right]$$

$$\left[\cosh \frac{\alpha}{2} + \sinh \frac{\alpha}{2} (\hat{\alpha} \cdot \vec{\sigma}) \right]$$

$$= \cosh^2 \frac{\alpha}{2} - \sinh^2 \frac{\alpha}{2} (\hat{\alpha}' \cdot \vec{\sigma})(\hat{\alpha} \cdot \vec{\sigma}) - \frac{1}{2} \sinh \alpha (\hat{\alpha}' - \hat{\alpha}) \cdot \vec{\sigma}$$

$$(\hat{\alpha}' \cdot \vec{\sigma})(\hat{\alpha} \cdot \vec{\sigma}) = (\hat{\alpha}' \cdot \hat{\alpha}) + i(\hat{\alpha}' \times \hat{\alpha}) \cdot \vec{\sigma}$$

$$\simeq 1 - i \Delta \theta \cdot \sigma_z$$



$$\Delta \theta (\hat{e}_\theta \cdot \vec{\sigma})$$

$$\Rightarrow K(\vec{\alpha}') K^{-1}(\vec{\alpha}) = 1 + i \sinh^2 \frac{\alpha}{2} \Delta \theta \cdot \sigma_z - \frac{1}{2} \sinh \alpha$$

$$= (1 - \frac{1}{2} \sinh \alpha \cdot \Delta \theta \hat{e}_\theta \cdot \vec{\sigma}) (1 + i \frac{1}{2} [\cosh \alpha - 1] \Delta \theta \sigma_z)$$

$$\leftarrow K \left[\sinh \alpha \cdot \Delta \theta \hat{e}_\theta \right] \cdot U_z \left(\underbrace{-(\cosh \alpha - 1) \cdot \Delta \theta}_{\text{this is a rotation around } \hat{z}\text{-axis}} \right)$$

this is a rotation around
 $\hat{z}$ -axis

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} = \cosh \alpha \Rightarrow \cosh \alpha - 1 = \gamma - 1$$

change to velocity

$$\Delta \theta = \frac{\vec{v} \times \vec{v}'}{v^2} = \frac{\vec{\beta} \times (\vec{\beta}' - \vec{\beta})}{\beta^2} \quad \text{and } \gamma^2 - 1 = \beta^2 \gamma^2$$

$$\text{Hence, the rotation angle } \Delta \Omega = - \frac{\gamma - 1}{\beta^2} (\vec{\beta} \times \delta \vec{\beta})$$

(7)

$$\Rightarrow \boxed{\Delta v_2 = - \frac{\gamma^2}{1+\gamma} (\vec{\beta} \times d\vec{\beta})}$$

$$\text{hence } \vec{\omega}_T = - \frac{\gamma^2}{1+\gamma} (\vec{\beta} \times \frac{d\vec{\beta}}{dt})$$

$$= - \frac{\gamma^2}{c^2(1+\gamma)} (\vec{v} \times \vec{a}) = \frac{\gamma^2}{1+\gamma} \frac{\vec{a} \times \vec{v}}{c^2}$$

$\nwarrow$ neglect correction at β^2 $\nwarrow$ already at v^2/c^2

we also neglect $\left(\frac{ds}{dz}\right)_m$ and $\left(\frac{ds}{dt}\right)_m$

For atomic problem

$$\begin{aligned} \omega_T &\approx \frac{1}{2c^2} \frac{\vec{F}}{m^2} \frac{\vec{r}}{r} \times m\vec{v} \\ &= - \frac{1}{2m^2c^2} \left(\frac{1}{r} \frac{dv}{dr} \right) \vec{L} \end{aligned}$$

$$\text{Hence } \boxed{\mathcal{U} = - \frac{eg}{2mc} \vec{S} \cdot \vec{B} + \frac{(g-1)}{2m^2c^2} (\vec{S} \cdot \vec{L}) \left(\frac{1}{r} \frac{dv}{dr} \right)}$$

Inside atom $\frac{dv}{dr} > 0$, hence the low energy state

corresponds to $j_- = l - 1/2$.
 $\swarrow$
 the branch of

(8)

However, as for the problem of nuclear physics,

$V_{IV}(r)$ is not due to electric field. In fact, the

EM part $V_{EM}(r)$ is repulsive, hence it contributes

$$\frac{g}{2m_n^2 c^2} (\vec{S} \cdot \vec{L}) \left(\frac{1}{r} \frac{dV_{EM}}{dr} \right) \rightarrow \frac{dV_{EM}}{dr} < 0.$$

Thomas precession $\rightarrow - \frac{1}{2m_n^2 c^2} (\vec{S} \cdot \vec{L}) \left(\frac{1}{r} \frac{d(V_{IV} + V_{EM})}{dr} \right)$

$$\Rightarrow U_{SO} = - \frac{1}{2m_n^2 c^2} (\vec{S} \cdot \vec{L}) \left(\frac{1}{r} \right) \left[(1+g) \frac{dV_{EM}}{dr} + \frac{dV_N}{dr} \right]$$

dV_N/dr dominates

