Lect 14 Theory of angular momentum (I)

§1 eigenvalues, eigenstates of angular momentum

$[J_\alpha, J_\beta] = i\varepsilon_{\alpha\beta\gamma} J_\gamma$, $D(g) = e^{-i\vec{n}\cdot\vec{J}\theta}$, for $g(\vec{n},\theta)$

we use $|j,m\rangle$ to represent the common eigenstates of

$J^2 = J_x^2 + J_y^2 + J_z^2$, and J_z, such that

$J^2|j,m\rangle = \lambda_j |jm\rangle$ and $J_z|jm\rangle = \lambda_m |jm\rangle$ below.

we will determine the above eigenvalues of $\lambda_j = j(j+1)$ and $\lambda_m = m$

Set $J_\pm = J_x \pm i J_y$, we have $J_\pm^\dagger = J_\mp$,

its easy to prove that $[J^2, J_\pm] = [J^2, J_z] = 0$.

Ex: check $[J_+, J_-] = 2J_z$ and $J^2 = J_+J_- + J_z(J_z-1)$
$= J_-J_+ + J_z(J_z+1)$

$J^2 J_\pm |jm\rangle = J_\pm J^2 |jm\rangle = \lambda_j J_\pm |jm\rangle$

$[J_z, J_\pm] = [J_z, J_x \pm i J_y] = iJ_y \pm i(-)iJ_x = \pm(J_x \pm iJ_y) = \pm J_\pm$

$J_z J_\pm |jm\rangle = (J_\pm J_z \pm J_\pm)|jm\rangle = (m \pm 1) J_\pm |jm\rangle$

Thus we can start from $|jm\rangle$. and reach

$J_+|jm\rangle, (J_+)^2|jm\rangle, \ldots (J_+)^k|jm\rangle$ whose eigenvalues of J_z are $m+1, \cdots m+k$.

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also $J_-|jm\rangle,\ (J_-)^2|jm\rangle,\ \cdots\ (J_-)^{k'}|jm\rangle,$ who eigenvalues of J_z are

$m-1,\ m-2,\ \cdots\ m-k'.$

We will show that these two sequences will terminate at finite lengthes.

This is because all the $(J_+)|jm\rangle, \cdots (J_+)^k|jm\rangle,\ J_-|jm\rangle, \cdots (J_-)^{k'}|jm\rangle$ share the same (eigen) value of J^2, ie. λ_j. $J^2 = J_x^2 + J_y^2 + J_z^2 \Rightarrow \lambda_j \geq J_z^2$,

thus $(m+k)^2, (m-k')^2 \leq \lambda_j$, $\Rightarrow$ k and k' must terminate at finite values.

Let us just assume such a sequence with both ends

$$\underbrace{(J_-)^{\underline{k}}|jm\rangle, \cdots J_-|jm\rangle}_{\underline{k}\text{ terms}},\ |jm\rangle,\ \underbrace{J_+|jm\rangle, \cdots (J_+)^{\bar{k}}|jm\rangle}_{\bar{k}\text{ terms}}$$

$(J_+)^{\bar{k}+1}|jm\rangle = 0$ we cannot further apply J_+ on $(J_+)^{\bar{k}}|jm\rangle$,

$(J_-)^{\underline{k}+1}|jm\rangle = 0$ and cannot apply J_- on $(J_-)^{\bar{k}}|jm\rangle$.

from $J^2 = J_-J_+ + J_z(J_z+1) = J_+J_- + J_z(J_z+1)$, we have

$$J^2\,(J_+)^{\bar{k}}|jm\rangle = \big[J_-J_+ + J_z(J_z+1)\big]\underbrace{|jm\rangle}_{(J_+)^{\bar{k}}} = (m+\bar{k})(m+\bar{k}+1)\Big\{(J_+)^{\bar{k}}|jm\rangle\Big\}$$

$$J^2\,[J_-]^{\underline{k}}|jm\rangle = \big\langle J_+J_- + J_z(J_z-1)\big](J_-)^{\underline{k}}|jm\rangle = (m-\underline{k})(m-\underline{k}-1)\{(J_-)^{\underline{k}}|jm\rangle\}$$

$$\Rightarrow\ \lambda_j^2 = (m+\bar{k})(m+\bar{k}+1) = (m-\underline{k})(m-\underline{k}-1)$$

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Because $\bar{k}$ and $\underline{k}$ are positive integers, we have

$$\left.\begin{aligned} m + \bar{k} &= -(m - \underline{k}) \\ m + \bar{k} + 1 &= -(m - \underline{k} - 1) \end{aligned}\right\} \Rightarrow \quad 2m = \underline{k} - \bar{k},$$

thus m can only take integer, or, half integer values.

Let $j = m + \bar{k} = -(m - \underline{k}) \Rightarrow J^2 = j(j+1)$.

Conclusion: For states $|jm\rangle$ satisfying

$$J^2|jm\rangle = j(j+1)|jm\rangle \quad \text{and} \quad J_z|jm\rangle = m|jm\rangle,$$

we have $-j \le m \le j$, and m, j can only be integer, or, half an integer.

$m = -j, -j+1, \cdots j$, takes $2j+1$ possible eigenvalues.

§ normalization and convention of relative phase of $|jm\rangle$.

Consider $|njm\rangle$ which represent a set of orthonormal complete bases for a system. n is another good quantum number, which represents another mechanical observable commutable with J^2, J_z.

$$J_\pm |njm\rangle = C_\pm |nj\,m\pm1\rangle$$

$$\Rightarrow |C_\pm|^2 = \langle njm| J_\mp J_\pm |njm\rangle = \langle njm| J^2 - J_z(J_z \pm 1)|njm\rangle$$

$$= j(j+1) - m(m\pm1) = (j \mp m)(j \pm m + 1)$$

we fix the phase convention that $C_\pm$ are real $\Rightarrow$

$$\boxed{J_\pm |njm\rangle = \sqrt{(j\mp m)(j\pm m+1)}\ |nj\,m\pm1\rangle}$$

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or $\langle n j m+1 | J_\pm | n j m \rangle = \sqrt{(j \mp m)(j \pm m+1)}$

$$|njm\rangle = (J_-)^{j-m} |njj\rangle \frac{\sqrt{(j+m)!}}{\sqrt{(2j)!}\sqrt{(j-m)!}}$$

$$= (J_+)^{j+m} |nj-j\rangle \sqrt{\frac{(j-m)!}{(2j)!\,(j+m)!}}$$

Important result:

① Assume that an operater K is rotationally invariant, i.e, $[K, \vec{J}]=0$

then it matrix element $\langle n' j m | K | n j m \rangle = f(n,j)$

is independent with m.

Proof. First of all, K is diagonal with respect to j, m.

$$J^2 K |njm\rangle = K J^2 |njm\rangle = j(j+1) K |njm\rangle$$

$$J_z K |njm\rangle = m K |njm\rangle$$

$\Rightarrow$ $K|njm\rangle$ shares the same eigenvalues as $|njm\rangle$ does.
of $j(j+1)$ and m

$\Rightarrow$ only $\langle n' jm | K | njm \rangle$ can be nonzero, i.e, it can only be non-diagonal with respect to n.

Second, $\langle n' j m+1 | K | n j m+1 \rangle = \dfrac{\langle n' jm | J_- K J_+ | njm \rangle}{(j-m)(j+m+1)}$

$$= \frac{\langle n' jm | K J_- J_+ | njm \rangle}{(j-m)(j+m+1)}$$

$$J_- J_+ = \vec{J}^2 - J_z(J_z+1) \quad \Rightarrow \quad J_- J_+ |njm\rangle = [j(j+1) - m(m+1)]\,|njm\rangle$$

$$= (j-m)(j+m+1)\,|njm\rangle$$

$$\Rightarrow \langle n'jm+1|\, K\, |njm+1\rangle = \langle n'jm|\, K\, |njm\rangle \qquad m = -j, \cdots j$$

$\Rightarrow \langle n'jm|K|njm\rangle$ is independent of m.

(2) Similarly, we can prove if there are two sets of angular momentum eigenstates $|\psi_{jm}\rangle$ and $|\Phi_{jm}\rangle$, we have $\langle \psi_{jm}|\Phi_{jm}\rangle$ is independent of m.

Later, we will see "j" is the quantum number to mark the representation of SU(2) group, and "$m = -j \cdots j$" is the label of the bases in such a representation. The above result is a specical case of Wigner-Eackart theorem, what states the above matrix elements are diagonal-blocked with respect to j, and proportional to identity matrix within each diagonal block.

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